Queen Mary
University of London

Main Examination period 2018

MTH743N / MTH743P / MTH743U: Complex Systems

Duration: 3 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: V. Latora, F. Vivaldi

Question 1. [31 marks]
Consider the map $f_r : [0,1] \to [0,1]$ defined piecewise by:

$$f_r(x) = \begin{cases} r\left(\sqrt{\left(\frac{2}{r^2}+\frac{4}{r}\right)x+1}-1\right) & \text{if} \quad 0 \le x \le 1/2 \\ r\left(\sqrt{\left(\frac{2}{r^2}+\frac{4}{r}\right)(1-x)+1}-1\right) & \text{if} \quad 1/2 < x \le 1 \end{cases}$$

where r is a tuning parameter that can take values in the range $(0,4]$.

(a) Show that f_r is invertible on each branch, by determining explicitly the two inverse functions. **[8]**

(b) Write down the Frobenius-Perron equation of the map f_r. **[6]**

(c) Consider the function

$$\rho_r(x) = C_r(x+r)$$

where C_r is a normalisation constant. Determine the dependence of C_r on r such that $\rho_r(x)$ is the density of a probability measure on $[0,1]$ for each value of r in $(0,4]$. **[4]**

(d) Show that $\rho_r(x)$ is a solution of the Frobenius-Perron equation of the map f_r. **[6]**

(e) Assuming that the density $\rho_r(x)$ gives rise to an ergodic invariant measure, compute the Lyapunov exponent of the map f_r in the case $r = 1/2$. Why is ergodicity important here? **[7]**

Question 2. [36 marks]
Consider the map $f : [0,1] \to [0,1]$ defined as

$$f(x) = \begin{cases} \frac{2}{3}(1+x) & \text{if} \quad 0 \le x \le \frac{1}{2} \\ 2(1-x) & \text{if} \quad \frac{1}{2} < x \le 1 \end{cases}$$

(a) Sketch the graph of the map. Find the fixed points and the orbits of period two of the map, and assess their linear stability. **[7]**

(b) Find a Markov partition and show that the map is an expanding Markov map. **[8]**

(c) Write down the topological transition matrix of the map and compute the number of periodic symbol sequences of period p, with $p = 1,2,3,4$. Write down all admissible periodic symbol sequences of period $p = 2$. Does the map have periodic points of period three? **[10]**

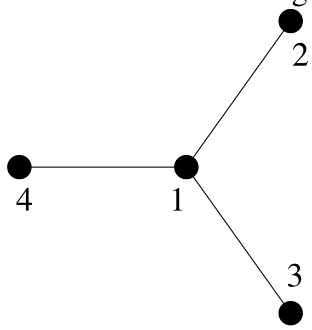
(d) Calculate the topological entropy of the map. **[4]**

(e) Determine the transfer matrix of the map. Find an expression for the invariant density and sketch the density in a diagram. **[7]**

Question 3. [33 marks]
Consider the following equations of motion:

$$\dot{x}_i(t) = f(x_i(t)) + \sigma \sum_{j}^{N} G_{ij} h(x_j(t)) \quad i = 1, 2, \ldots, N$$

describing the dynamics of a coupled network of N nodes, where $x_i(t)$ denotes the state at node i, $f(x) = x(1 - x^2)$ governs the local node dynamics, $h(x)$ determines the form of the coupling, $\sigma \in \mathbb{R}$ is the coupling strength, and G_{ij} is the Laplacian of the underlying graph. The network consists of $N = 4$ nodes connected as in the graph below:



Consider the diffusive coupling $h(x) = h_1(x) = x$.

a) Determine the time-independent synchronised states. **[5]**

b) For each of the time-independent synchronised states, compute the master stability function. **[8]**

c) Define the Laplacian of a network; hence determine the eigenvalues of the Laplacian of this network. **[6]**

d) For each synchronised state find the values of the coupling strength σ such that the state is transversely stable. **[8]**

e) Instead of the diffusive coupling $h(x) = h_1(x) = x$, consider now the coupling function $h(x) = h_2(x) = 1/(2 + x)$. For each synchronised state, find the new values of the coupling strength σ such that the state is transversely stable. **[6]**

End of Paper.