M. Sci. Examination by course unit 2015

MTH743U: Complex Systems

Duration: 3 hours

Date and time: 26th May 2015, 14:30–17:30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): Prof. Vito Latora

Question 1 (30 marks).
Consider the map $f_\beta : [0,1] \to [0,1]$ defined by

$$f_\beta(x) = \begin{cases} \frac{x}{\beta} & \text{if} \quad 0 \le x \le \beta \\ \frac{1-x}{1-\beta} & \text{if} \quad \beta < x \le 1 \end{cases} .$$

where $0 < \beta < 1$.

(a) Find the fixed points of the map and determine their linear stability. **[6]**

(b) Find the two invertible branches of the map and write down the two inverse functions. **[6]**

(c) Write down the Frobenius-Perron equation of the map. **[7]**

(d) Find the invariant density of the map. **[6]**

(e) Assuming that the density $\rho(x)$ gives rise to an ergodic invariant measure, compute the probability that the points of a trajectory are in the interval $[\beta, 1]$. **[5]**

Question 2 (35 marks).
Consider the map $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \le x < 1/2 \\ 2x - 1 & \text{if} \quad 1/2 \le x < 3/4 \\ 2x - 3/2 & \text{if} \quad 3/4 \le x \le 1 \end{cases}$$

(a) Sketch the graph of the map f, and show that the map is piecewise linear. **[7]**

(b) Show that the partition $\{I_0, I_1, I_2\}$ with $I_0 = [0, 1/2]$, $I_1 = [1/2, 3/4]$ and $I_2 = [3/4, 1]$ is a Markov partition, and write down the corresponding topological transition matrix. **[5]**

(c) Show that the map f is expanding. Find the number M_p of admissible periodic symbol sequences of period $p = 1$, $p = 2$, and $p = 5$. Determine all admissible periodic symbol sequences of period $p = 1$ and $p = 2$. Do they correspond to periodic points, respectively of period $p = 1$ and $p = 2$, of the map? Justify your answer. **[11]**

(d) State the definition of the topological entropy of a map, and explain why the topological entropy is a measure of the complexity of a system. Evaluate the topological entropy of the map f. **[6]**

(e) Determine the transfer matrix of the map f and compute its invariant density. Assuming ergodicity, evaluate the Lyapunov exponent of the map f. **[7]**

Question 3 (35 marks).
Consider a network of $N = 4$ fully-coupled logistic equations described by the equations:

$$\dot{x}_i(t) = x_i(t)(r - x_i(t)) + \sigma \sum_{j=1}^{N} (x_i(t) - x_j(t)) \quad i = 1, \ldots, N$$

where $x_i(t)$ denotes the state (population) of node i, $r \in \mathbb{R}^+$ is the so-called carrying capacity of each node, and $\sigma \in \mathbb{R}$ denotes the coupling strength.

(a) Rewrite the equations in terms of the adjacency matrix A_{ij} of the underlying graph. Rewrite the equations in terms of the Laplacian matrix G_{ij} of the underlying graph. **[8]**

(b) Determine the time-independent synchronised states. Compute the master stability function for the time-independent synchronised states. **[8]**

(c) For the non-trivial time-independent synchronised state, find the values of σ such that the state is transversely stable. **[6]**

(d) How many different interaction networks can you obtain by removing two links from the original graph? Label the nodes, and write the corresponding adjacency matrices, and the degree distributions of the resulting graphs. **[6]**

(e) For the non-trivial time-independent synchronised state, find the values of σ such that the state is transversely stable on the new graphs. Do things change with respect to the case considered in (c)? **[7]**

End of Paper.