Queen Mary University of London

Main Examination period 2019

MTH734U/MTH712P: Topics in Probability and Stochastic Processes

Duration: 3 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Only non-programmable calculators that have been approved from the college list of non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: A. Gnedin, D. Ellis

Question 1. [20 marks] Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathbb{P}(X_j = 1) = p$, $\mathbb{P}(X_j = -1) = 1 - p = q$ for some $0 < p < 1$. Consider a simple random walk $(S_n,\ n \geq 0)$, with $S_0 = 0$ and $S_n = \sum_{j=1}^{n} X_j$ for $n \geq 1$. A standard result on random walks, which you can use without proof, is given in the Appendix.

(a) For each given p such that $0 < p < 1$, what is the probability that the random walk ever reaches state 2? Explain your answer. **[6]**

(b) For $p = 1/2$, what is the probability that the random walk eventually reaches both the state 2 and the state -2, with the first visit to the state -2 occurring before the first visit to the state 2? Explain your answer. **[3]**

(c) Let $\tau = \min\{n \geq 0 : S_n = 2\}$ be the time of the first visit to state 2.

  (i) Explain what it means for this random variable τ to be a stopping time adapted to $X_1, X_2, \ldots$ **[2]**

  (ii) Assuming that $p > 1/2$, find $\mathbb{E}\tau$. **[4]**

(d) Suppose $p = 1/3$. Show that as $n \to \infty$, with probability one

$$\frac{S_n - 3S_{n+1}}{2n} \to \frac{1}{3}\,.$$

You may use any standard result from the course, without proof. **[5]**

Question 2. [20 marks]

(a) Let $(N(t),\ t \geq 0)$ be a Poisson process with rate λ. State without proof which of the following is true for positive h as $h \to 0$:

(i) $\mathbb{P}(N(h) = 0) = o(h)$, **[1]**

(ii) $\mathbb{P}(N(h) = 1) = \lambda h + o(h)$, **[1]**

(iii) $\mathbb{P}(N(h) = 2) = o(h)$, **[1]**

(iv) $\mathbb{E}[N(h)] = \lambda h + o(h)$. **[2]**

(b) A counter registers two types of particles, A and B, that arrive according to Poisson processes with rates 2 and 3 particles per second, respectively. The Poisson processes for particles A and B are independent of one another. In the following questions (i)-(v), state any properties of the Poisson process, that you use.

(i) What is the distribution of the total number of particles (of any type) that arrive in the first two seconds? **[3]**

(ii) Given that exactly one particle arrived before time t, what is the distribution of the arrival time of this particle? **[3]**

(iii) Given that five particles arrived before time t, what is the distribution of the number of type A particles among them? **[3]**

(iv) Given that three particles of type A and one particle of type B arrived before time t, find the expected further waiting time (after time t) for the next particle of type A to arrive? **[3]**

(v) What is the probability that two particles of type A arrive within the time interval $[0, 1]$ and three particles of type B arrive within the time interval [4, 5] (with time measured in seconds)? **[3]**

Question 3. [20 marks] A processor executes without interruption an unending sequence of tasks, one task at a time. The durations of tasks are independent, with the same probability density function

$$f(x) = \begin{cases} 3x^{-4}, & \text{for } x \geq 1, \\ 0, & \text{for } x < 1. \end{cases}$$

The cost incurred by a task of duration $x \geq 0$ is given by the formula $y = 30\sqrt{x} + 10$. Assuming that the first task starts at time zero, let $N(t)$ and $Y(t)$ be the number and the cumulative cost of tasks completed by time $t \geq 0$, respectively.

(a) Let S_n be the time at which the nth task is completed. Show that for all $t \geq 0$ and integers $n \geq 1$

$$\mathbb{P}(N(t) = n) = \mathbb{P}(S_n \leq t < S_{n+1}).$$

[4]

(b) Find the long run task completion rate

$$\lim_{t\to\infty} \frac{N(t)}{t}.$$

[5]

(c) State a Central Limit Theorem for the number of completed tasks $N(t)$. Evaluate all parameters involved. **[6]**

(d) Find the long run mean cost per unit time

$$\lim_{t\to\infty} \frac{\mathbb{E}Y(t)}{t}.$$

[5]

Question 4. [20 marks] A network operates with two servers A and B. Each server switches at random times between online and offline modes. The length of every online period is exponentially distributed with mean 20 minutes, and the length of every offline period is exponentially distributed with mean 12 minutes. The lengths of periods are all independent. Let $X(t)$ be the number of servers in the online mode at time $t \geq 0$.

(a) Write down the generator matrix G of the continuous-time Markov chain $(X(t),\ t \geq 0)$. **[7]**

(b) Is the Markov chain $(X(t),\ t \geq 0)$ irreducible? Justify your answer briefly. **[3]**

(c) Determine the stationary distribution of the Markov chain $(X(t),\ t \geq 0)$. **[7]**

(d) For large t, what is the approximate probability that the server A is in the online mode at time t? **[3]**

Question 5. [20 marks] Let $(B(t),\ t \geq 0)$ be a (standard) Brownian motion.

(a) For $0 \leq s < t$, determine the distribution of $2B(s) + B(t)$. **[5]**

(b) For $0 \leq s < t$, what is the conditional distribution of $B(t)$ given $B(s) = x$? **[5]**

(c) Is the process

$$X(t) = \frac{1}{3}B(9t),\ t \geq 0,$$

a Brownian motion? Explain your answer. **[5]**

(d) Let $M(t) = \max_{s \in [0,t]} B(s)$ be the maximum of the Brownian motion on the time interval $[0, t]$. State without proof which of the following is true for fixed $x > 0$ and $t > 0$:

(i) $\mathbb{P}(M(t) > x) = \mathbb{P}(B(t) > x)$, **[1]**

(ii) $\mathbb{P}(M(t) > x) = 2\mathbb{P}(B(t) > x)$, **[1]**

(iii) $\mathbb{P}(|B(t)| > x) = 2\mathbb{P}(B(t) > x)$, **[1]**

(iv) $\mathbb{P}(M(t) > x, B(t) < x) = \mathbb{P}(B(t) > x)$. **[2]**

End of Paper – An appendix of 1 page follows.

Appendix: The Gambler's Ruin Problem

For integers $a > 0,\ b \geq 0$ the probability that a simple (p, q)-random walk starting at 0 and with probability p of moving to the right visits $-a$ before (if ever) reaching b is

$$\begin{cases} \frac{1-\left(\frac{p}{q}\right)^{b}}{1-\left(\frac{p}{q}\right)^{a+b}}, & \text{for } p \neq q, \\ \frac{b}{a+b}, & \text{for } p = q = 1/2. \end{cases}$$

The probability that the random walk visits b before (if ever) reaching $-a$ is one minus the expression above.

End of Appendix.