Queen Mary
University of London

M. Sci. Examination by course unit 2014

MTH734U: Topics in Probability and Stochastic Processes

Duration: 3 hours

Date and time: 2nd May 2014, 10:00–13:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): Dudley Stark

Question 1 Let $N(t)$, $t \geq 0$, be a continuous time renewal process with interoccurrence times $X_i > 0$ which are independent, identically distributed continuous random variables with common distribution $\mathbb{P}(X_i \leq x) = F(x)$. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the waiting time until the occurrence of the nth event. Let $M(t) = \mathbb{E}(N(t))$.

(a) State what is meant by the delayed renewal process $N_D(t)$ determined by the X_i and another probability distribution $G(x)$. [4]

(b) If the X_i are Uniform$(0,1)$ distributed, then what is $\lim_{t\to\infty} \frac{M(t)}{t}$? [3]

(c) Prove that $M(t) = \sum_{k=1}^{\infty} \mathbb{P}(S_k \leq t)$. [5]

(d) (i) Let $f(x)$ denote the probability density function of the X_i. Prove that $\hat{f}(\lambda) = \lambda \hat{F}(\lambda)$, where $\hat{f}(\lambda)$ and $\hat{F}(\lambda)$ denote the Laplace transforms of $f(x)$ and $F(x)$, respectively. [4]

(ii) Using the integral equation

$$M(t) = \int_0^t M(t-x) f(x)\, dx + F(t),$$

derive the equation

$$\hat{M}(\lambda) = \frac{\hat{F}(\lambda)}{1 - \lambda \hat{F}(\lambda)}$$

where $\hat{M}(\lambda)$ denotes the Laplace transform of $M(t)$. You may use properties of Laplace transforms proved in lectures or problem sheets, but you must state explicitly what properties you are using. [4]

Question 2

(a) Given a semi-Markov process on states $\{1, 2, \ldots, N\}$, suppose that when the process enters state i, it stays there a random amount of time having expectation μ_i after which it jumps to state j with probability $P_{i,j}$.

(i) State what is meant by the Markov chain associated to the semi-Markov process. [3]

(ii) Suppose that $N = 3$, $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 3$, $P_{1,2} = P_{2,3} = P_{3,1} = 1$. Let the quantities P_i denote the proportion of time in the long run that this semi-Markov process is in state i. Find the P_i. [6]

(b) The light bulbs used in an industrial process have life times which are independent and Uniform$(0, 300)$ distributed, with time measured in days. Suppose that a light bulb is replaced immediately upon its failure or after 100 days of use, whichever comes first. It costs £2 to replace a light bulb before 100 days of use and only £1 to replace a light bulb after exactly 100 days of use. What is the long-run cost of replacing light bulbs per unit time? [11]

Question 3

(a) (i) State the definition for a function to be in the class $o(h)$. [2]

(ii) Let $N(t)$ be a Poisson process with rate θ. Prove that

$$\mathbb{P}(N(h) = 1) = \theta h + o(h).$$

[4]

(b) A radioactive source emits particles according to a Poisson process with rate two particles per minute.

(i) What is the probability that the first particle appears after three minutes? [2]

(ii) What is the probability that the first particle appears after three minutes but before five minutes? [4]

(iii) What is the probability that exactly one particle is emitted in the interval from three to five minutes? [2]

(iv) What is the probability that exactly one particle is emitted in the interval from zero to four minutes and that exactly one particle is emitted in the interval from three to five minutes? [6]

Question 4

(a) Let $X(t)$ be a continuous time Markov chain with conditional probability densities

$$f_n(y_n, t_n | y_{n-1}, t_{n-1}; y_{n-2}, t_{n-2}; \ldots ; y_1, t_1),$$

where $0 \le t_1 < t_2 < \cdots < t_n$ and $y_i \in \mathbb{R}$ for all $1 \le i \le n$, where $\mathbb{R}$ is the set of real numbers. State what is meant by the Markov property for $X(t)$. [3]

(b) A system consists of two machines and one repairman. All machines in operating condition are operated. The amount of time in hours that an operating machine works before breaking down is exponentially distributed with mean 20. The amount of time in hours that it takes the repairman to fix a machine is exponentially distributed with mean 1. The repairman can work on only one failed machine at any given time. Let $X(t)$ be the number of machines in operating condition at time t.

(i) Find the generator for $X(t)$. [9]

(ii) Calculate the long run probability distribution for $X(t)$. [6]

(iii) In the long run, what is the proportion of time that the repairman is fixing a machine? [2]

Question 5 Let $B(t)$ be standard Brownian motion with $B(0) = 0$.

(a) Given $t > 0$, state the distribution of $B(t)$, including any parameters. [2]

(b) Given real numbers $0 \leq s < t$, determine the joint probability density function

$$f_{B(s),B(t)}(x_1, x_2).$$

[4]

(c) Define the process $X(t)$ by

$$X(t) = e^{-\alpha t/2} B(e^{\alpha t}).$$

(i) Show that $\mathrm{Cov}(X(s), X(t))$ depends only on $|s - t|$. [6]

(ii) Is $X(t)$ Brownian motion? Briefly justify your answer. [2]

(d) Let $M(t) = \max_{0 \leq s \leq t} B(s)$. Let $a > 0$ and $y \geq 0$. Use the Reflection Principle to show that

$$\mathbb{P}(B(t) \leq a - y, M(t) \geq a) = \int_{(a+y)/\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

[6]

End of Paper