Queen Mary
University of London

Main Examination period 2017

MTH714U / MTHM024: Group Theory

Duration: 3 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: L. H. Soicher and A. R. Fink

Question 1. Let G be a group and let Ω be a set, with $|\Omega| > 1$.

(a) Suppose G acts transitively on Ω. What is meant by a G-**congruence** on Ω? What does it mean to say that G acts **primitively** on Ω, and what does it mean to say that G acts **doubly transitively** on Ω? [**6**]

(b) Prove that if G acts doubly transitively on Ω then G acts primitively on Ω. [**5**]

(c) What does it mean for a permutation of $\{1, \ldots, n\}$ to be **even**, and what is meant by the **alternating group** A_n? [**4**]

(d) Prove that if $n \geq 4$ then the alternating group A_n acts doubly transitively on $\{1, \ldots, n\}$. [You do not need to prove that A_n is a group.] [**5**]

(e) Give an explicit example of a group H acting doubly transitively on a set S of size n, such that $|H| < n!/2$. [You should briefly justify your answer.] [**5**]

Question 2.

(a) What is meant by a **Sylow** p**-subgroup** of a finite group G? [**3**]

(b) State all parts of Sylow's theorems on the existence and properties of Sylow p-subgroups. [**6**]

(c) For each $p \in \{2, 3, 5\}$, determine explicitly a Sylow p-subgroup of the alternating group A_5. [You do not have to justify your answers.] [**3**]

(d) Let G be any simple group of order 60.

(i) Apply Sylow's theorems to prove that G is isomorphic to a subgroup of the symmetric group S_6. [**6**]

(ii) Then prove that G is isomorphic to A_5. [You may assume that A_6 is simple and is the only subgroup of index 2 in S_6.] [**7**]

Question 3. Let G be a group.

(a) Define what is meant by an **automorphism** of G, and what is meant by an **inner automorphism** of G. **[4]**

(b) Assuming that the set $\mathrm{Aut}(G)$ of all automorphisms of G forms a subgroup of $\mathrm{Sym}(G)$, and that each inner automorphism of G really is an automorphism of G, prove that the set of inner automorphisms of G is a normal subgroup of $\mathrm{Aut}(G)$. **[8]**

(c) Suppose now that G is a non-trivial finite group, such that the group $\mathrm{Aut}(G)$ (in its natural action on G) acts transitively on the set of non-identity elements of G.

(i) Prove that each non-identity element of G has the same order p, for some prime p, and so deduce that G is a group of order p^a, for some integer $a > 0$. [You may make use of any results proved in the lectures.] **[7]**

(ii) Prove that G is abelian. [You may make use of any results proved in the lectures.] **[6]**

Question 4. Suppose n is an integer greater than 1 and that F is a field.

(a) Define the groups $\mathrm{GL}(n, F)$, $\mathrm{SL}(n, F)$, and $\mathrm{PSL}(n, F)$. **[6]**

(b) Give, without proof, the orders of the above groups, in the case where F is the finite field $\mathbb{F}_q$. **[3]**

(c) Explain why $\mathrm{PSL}(2, 2) \cong S_3$ and $\mathrm{PSL}(2, 3) \cong A_4$. **[4]**

(d) Describe briefly and precisely the main steps of the proof that $\mathrm{PSL}(n, F)$ is simple, except in the two cases $n = 2, F = \mathbb{F}_2$ and $n = 2, F = \mathbb{F}_3$. **[12]**

End of Paper.