Queen Mary
University of London

M. Sc. Examination by course unit 2014

MTHM024 Group Theory

Duration: 3 hours

Date and time: 6 May 2014, 10.00h–13.00h

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You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 4 questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): L. H. Soicher

Question 1 Let G be a group and let H and K be subgroups of G.

(a) Prove that $H \cap K$ is a subgroup of G, and furthermore, that if K is a normal subgroup of G then $H \cap K$ is a normal subgroup of H. [5]

(b) Prove that if $H \not\subseteq K$ and $K \not\subseteq H$, then $H \cup K$ is **not** a subgroup of G. [5]

(c) Prove that if K is a normal subgroup of G then HK is a subgroup of G containing K. [5]

(d) State the First Isomorphism Theorem (for groups). [4]

(e) Apply the First Isomorphism Theorem to prove that if K is a normal subgroup of G then

$$H/(H \cap K) \cong HK/K.$$

[6]

Question 2 (a) What is meant by a *permutation* of $\{1,\dots,n\}$, what is meant by an *even* permutation of $\{1,\dots,n\}$, and what is meant by the *alternating group* A_n? What does it mean to say that a group G is *simple*? [8]

(b) Suppose G is a group acting primitively on a set Ω, and let N be a normal subgroup of G. Prove that either N acts trivially on Ω (that is, N lies in the kernel of the action), or N acts transitively on Ω. [7]

(c) Prove that the alternating group A_n is simple, for all $n \geq 5$. [You may assume, without proof, that the group A_n acts primitively on $\{1,\dots,n\}$, and that A_5 is simple.] [10]

Question 3 (a) What is meant by a *Sylow p-subgroup* of a finite group G? [3]

(b) State all parts of Sylow's theorems on the existence and properties of Sylow p-subgroups. [6]

(c) Let G be a non-abelian finite simple group having exactly n Sylow p-subgroups for some prime p dividing $|G|$. Prove that $|G|$ divides $n!$. Further, prove that $|G|$ divides $n!/2$. [10]

(d) Prove that there is no simple group of order 300. [6]

Question 4 Let G be a group.

(a) Define what is meant by an *automorphism* of G, what is meant by the *automorphism group* $\mathrm{Aut}(G)$ of G, and what is meant by the *centre* $\mathrm{Z}(G)$ of G. [6]

(b) Suppose that G is a group of order p^a for some prime p and some integer $a > 0$. Prove that $|\mathrm{Z}(G)| > 1$. [6]

(c) Suppose now that G is a non-trivial finite group, such that the group $\mathrm{Aut}(G)$ (in its natural action on G) acts transitively on the set of non-identity elements of G.

(i) Prove that each non-identity element of G has the same order p, for some prime p, and so deduce that G is a group of order p^a, for some integer $a > 0$. [7]

(ii) Prove that G is abelian. [6]

Question 5 (a) Let G be a group. Define what is meant by the commutator $[g,h]$ of elements $g,h \in G$, and what is meant by the *commutator subgroup* (or *derived group*) G' of G. [4]

(b) Let F be a field, let $n > 1$, let $V = F^n$, let a be a non-zero vector in F^n, and let $f : V \to F$ be a linear map with $af = 0$.

(i) Define what is meant by the *transvection* $T(a, f)$ on V, and what is meant by the *transvection group* $A(a)$. What is meant by $\mathrm{SL}(n,F)$? [6]

(ii) Explain why $A(a)$ can be considered to be a subset of $\mathrm{SL}(n,F)$, and then why $A(a)$ is an abelian subgroup of $\mathrm{SL}(n,F)$. [8]

(iii) Let $w = (0,1) \in F^2$. Prove that if $|F| > 3$ then $A(w) \leq \mathrm{SL}(2,F)'$. [7]

End of Paper