Queen Mary
University of London

Main Examination period 2023 – January – Semester A

MTH6141 / MTH6141P: Random Processes

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

For actuarial students only: This module also counts towards IFoA exemptions. For your submission to be eligible, **you must submit within the first 3 hours.**

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: Robert Johnson, Mark Walters

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Question 1 [25 marks]. Let $(X_0, X_1, X_2, \ldots)$ be the discrete-time, homogeneous Markov chain on state space $S = \{1, 2, 3, 4, 5\}$ with $X_0 = 3$ and transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 1/2 & 0 & 1/4 \\ 0 & 1/6 & 1/6 & 1/6 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Calculate $\mathbb{P}(X_{10} = 2 \mid X_8 = 4)$. **[3]**

(b) Calculate the probability that the chain is absorbed in state 5. **[5]**

(c) Calculate $\mathbb{E}\left(\sum_{t=0}^{T-1} X_t\right)$ where T is the time of absorption. **[5]**

(d) Explain why this chain has infinitely many equilibrium distributions. **[4]**

(e) Show that the first return probability for state 3 satisfies **[4]**

$$\frac{1}{6} \leq f_3 \leq \frac{1}{2}$$

(f) Let M_3 be the number of visits to state 3 (including the one at time 0). Use part (e) to determine upper and lower bounds on $\mathbb{E}(M_3)$. **[4]**

Question 2 [25 marks]. Let $(X(t) : t \geq 0)$ be a Poisson process of rate 2.

(a) For each of the following, write down a random variable related to this process which has the given properties:

- (i) Has the Po(4) distribution and is independent of $X(4)$, **[3]**
- (ii) Has the Po(4) distribution and is not independent of $X(4)$, **[3]**
- (iii) Has the Bin$(10, \frac{1}{3})$ distribution. **[3]**

(b) Consider the two intervals $I_1 = [0, 2], I_2 = [1, 3]$. Let U be the random variable which counts how many of the intervals I_1, I_2 contain an arrival from the process $X(t)$. Determine the probability mass function of U. **[4]**

(c) Form a new process by deleting alternate arrivals from the process $(X(t) : t \geq 0)$ starting with the first (so we keep the 2nd, 4th, 6th etc.). Let $(Y(t) : t \geq 0)$ be the process which counts surviving arrivals.

- (i) Calculate $\mathbb{P}(Y(t) = 0)$. **[4]**
- (ii) Determine the cumulative distribution function of the time of the first arrival in the process $(Y(t) : t \geq 0)$. **[4]**
- (iii) Explain why this shows that $(Y(t) : t \geq 0)$ is not a Poisson process. **[2]**

(iv) What is wrong with the following argument: **[2]**

> The process $(Y(t):\ t \geq 0)$ is a thinned version of the process $(X(t):\ t \geq 0)$ in which each arrival is deleted with probability $\frac{1}{2}$. Therefore, by the Thinning Lemma, $(Y(t):\ t \geq 0)$ is a Poisson process of rate 1.

Question 3 [25 marks]. Let $(X_0, X_1, X_2, \ldots)$ be the discrete-time, homogeneous Markov chain on state space $S = \{1, 2, 3, 4, 5, 6\}$ with $X_0 = 1$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1/4 & 3/4 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find the unique equilibrium distribution for this Markov chain. **[5]**

(b) Does this Markov chain have a limiting distribution? Justify your answer. **[3]**

(c) What can you say about the proportion of time that the chain spends in state 1? **[3]**

(d) Calculate $\mathbb{P}(X_t = 1)$ for all t. **[3]**

Let $(Y_0, Y_1, Y_2, \ldots)$ be a new discrete-time homogeneous Markov chain on the same state space obtained by modifying this chain as follows. At each time I toss a biased coin which has probability $1/3$ of showing Heads. If it shows a Head then I take one step in the chain with transition matrix P; if it shows a Tail I remain in the same state. Let Q be the transition matrix for $(Y_0, Y_1, Y_2, \ldots)$.

(e) Write down the matrix Q. **[4]**

(f) Does this Markov chain have a limiting distribution? Justify your answer. **[3]**

(g) What can you say about the probability that the chain is in an odd numbered state at time t? **[4]**

Question 4 [25 marks].

(a) The integers from 1 to 7 are written in order clockwise around a circle. I start with a counter on number 1 and repeatedly roll a standard six-sided fair die. If the die shows 1 then I move the counter to the next number clockwise around the circle; if the die shows 2 or 3 then I move the counter to the next number anticlockwise around the circle; if the die shows $4, 5$ or 6 then I do not move the counter.

(i) Describe how to model this process as a discrete-time Markov chain. Write down the transition matrix for the chain. **[5]**

(ii) Explain briefly why the Markov property is satisfied. **[3]**

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(iii) Give an example of how you could make a small modification after which the process would be a discrete-time Markov chain which is not homogeneous. [**2**]

(b) A machine is running continuously except when it is broken. Suppose that while the machine is running, breakages happen according to a Poisson process of rate α. As soon as the machine breaks an engineer is called. The time it takes for the engineer to arrive follows an $\mathrm{Exp}(\beta)$ distribution. When the engineer has arrived they spend a time with $\mathrm{Exp}(\gamma)$ distribution working on the machine. At the end of this time it is either mended and running (this happens with probability p) or permanently broken (this happens with probability $1 - p$).

(i) Describe how to model the state of the machine as a continuous-time Markov chain with four states. Write down the generator matrix for the chain. [**5**]

(ii) Suppose that the machine is running at time 0. What is the probability that it runs continuously without breaking until at least time 1? [**3**]

(iii) Why is it natural to model this situation as a continuous-time process. [**2**]

(c) Which result about continuous-time processes from this module appeals to you the most? Give an informal indication of what this result says. Say what it is about this that you particularly like and why. [**5**]

(I expect your answer to part (c) to contain roughly three to five sentences. It should consist mainly of coherent written English rather than mathematical symbols.)

End of Paper.