MTH6141: Random Processes

Duration: 2 hours

Date and time: 10th May 2016, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): David Ellis

Question 1. Parts (a)-(c) of this question are about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Draw a transition graph for this Markov chain. [**3**]

(b) List the absorbing states of the Markov chain. [**1**]

(c) Using first-step analysis, or otherwise, find the probability that the Markov chain is eventually absorbed in state 5, given that $X_0 = 1$. Show your working. [**6**]

Parts (d)-(g) of this question are about an experiment. I have two urns and three balls: one red ball and two yellow balls. At the start of the experiment, all three of the balls are in one of the urns (the other urn is empty). Every minute, I pick one of the three balls at random (each is chosen with probability $\frac{1}{3}$), I take it out of the urn it is currently in, and I place it in the other urn. I stop (and the experiment ends) when both the yellow balls are in one urn and the red ball is in the other.

(d) This experiment can be modelled as a Markov chain with six states (of which two are absorbing states). Write down what these six states are. (In other words, write down which physical situations each of the six states corresponds to.) [**3**]

(e) Draw a transition graph for this Markov chain. [**4**]

(f) Write down a transition matrix for the Markov chain. (Indicate clearly which rows of the matrix correspond to which states.) [**2**]

(g) Using first-step analysis, or otherwise, find the expected length of time until the experiment ends. Show your working. [**6**]

Question 2. Parts (a) and (b) of this question are about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space $S = \{1, 2, \ldots, n\}$ and transition matrix P.

(a) Let $w = (w_1, w_2, \ldots, w_n)$ be a probability vector. Define what it means for w to be an *equilibrium distribution* for the Markov chain, and define what it means for w to be a *limiting distribution* for the Markov chain. [**4**]

(b) Prove that if the Markov chain is irreducible and has $p_{i,i} > 0$ for some state i, then it is regular. [**4**]

Parts (c) and (d) of this question are about a Markov chain $(Y_0, Y_1, Y_2, \ldots)$ with state space $\{1, 2, 3, 4\}$ and transition matrix

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

(c) Is this Markov chain regular? Justify your answer. [**3**]

(d) Find the limiting distribution of the Markov chain. Justify your answer. [**5**]

Parts (e)-(g) of this question are about a Markov chain $(Z_0, Z_1, Z_2, \ldots)$ with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \end{pmatrix}.$$

(e) Is this Markov chain irreducible? Justify your answer carefully. [**3**]

(f) Find the equilibrium distribution of the Markov chain. [**6**]

(g) Show that the Markov chain has no limiting distribution. [**5**]

Question 3. Parts (a) and (b) of this question are about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space S.

(a) Define what it means for a state $i \in S$ to be *recurrent*, in terms of the return probability $f_{i,i}$. **[2]**

(b) State a condition, in terms of the t-step transition probabilities $p_{i,i}^{(t)}$ alone, which is equivalent to the state i being recurrent. **[2]**

Parts (c)-(e) of this question are about the symmetric random walk on $\mathbb{Z}$, i.e. the Markov chain with state space $\mathbb{Z}$ and transition probabilities

$$p_{i,i+1} = \tfrac{1}{2} \text{ for all } i \in \mathbb{Z}, \quad p_{i,i-1} = \tfrac{1}{2} \text{ for all } i \in \mathbb{Z}, \quad p_{i,j} = 0 \text{ for all other } i, j \in \mathbb{Z}.$$

(c) Show that

$$p_{0,0}^{(t)} = \begin{cases} \binom{t}{t/2} 2^{-t} & \text{if } t \text{ is even;} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

[4]

(d) Using part (b) and part (c), or otherwise, show that the state 0 is recurrent. You may assume that

$$\binom{2n}{n} \geq \frac{2^{2n}}{2\sqrt{2n}} \quad \forall n \in \mathbb{N},$$

and that the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

diverges for all real numbers $s \leq 1$. **[3]**

(e) Is the state 99 recurrent or transient? Justify your answer briefly. **[2]**

Question 4. Buses arrive at a bus stop according to a Poisson process of rate 4 per hour. Your answers to the following questions should be expressed in powers of e (where appropriate), but they should be simplified in all other ways.

(a) What is the probability that no bus arrives between 8:00 am and 8:30 am? **[2]**

(b) Given that no bus arrives between 8:00 am and 8:30 am, what is the probability that at least two buses arrive between 8:30 am and 9:00 am? **[3]**

(c) Each bus which arrives at the bus stop is 'out of service' with probability 1/4, independently of all the other buses. What is the probability that at least two 'in service' buses arrive between 9:00 am and 10:00 am? **[4]**

(d) Given that exactly four buses arrive between 10:00 am and 11:00 am, what is the probability that exactly two arrived between 10:00 am and 10:20 am? [**4**]

(e) Suppose I arrive at the bus stop at 11:00 am. Let T_2 denote the time (in hours) I have to wait before I have seen a total of two buses arrive. Find (with justification) the probability density function of T_2. [**5**]

Question 5. (a) Let $\lambda_0, \lambda_1, \lambda_2, \ldots$ be non-negative real numbers. Let $(X(t):\ t \geq 0)$ be a continuous-time stochastic process with state space $\mathbb{N} \cup \{0\}$. Define what it means for $(X(t):\ t \geq 0)$ to be a *birth process* with birth parameters $(\lambda_0, \lambda_1, \lambda_2, \ldots)$. [**5**]

(b) Suppose that a population of yeast cells has one individual at the start of an experiment, and each individual in the population gives birth to offspring according to a Poisson process of rate 2 per minute (with these Poisson processes being independent of one another). Let $X(t)$ denote the number of individuals in the population at time t minutes after the start of the experiment. What are the birth parameters of the birth process $(X(t):\ t \geq 0)$ in this case? [**2**]

Parts (c)-(d) of this question are about a birth process $(X(t):\ t \geq 0)$ with birth parameters given by

$$\lambda_0 = 0,\ \ \lambda_1 = 2,\ \ \lambda_n = 4\ \forall n \geq 2,$$

and with initial state $X(0) = 1$. For each $n \in \mathbb{N}$, we define the function $p_n(t) = \text{Prob}(X(t) = n)$. Recall that the functions $p_n(t)$ satisfy the differential equations

$$p_n'(t) = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t),$$

for all $n \in \mathbb{N}$.

(c) Use these differential equations to find an explicit formula for $p_1(t)$ as a function of t. [**2**]

(d) Use them to find an explicit formula for $p_2(t)$ as a function of t. [**5**]

End of Paper.