Queen Mary University of London

Main Examination period 2022 – January – Semester A

MTH6140: Linear Algebra II

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: S. Majid, O. Jenkinson

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In your answers, all expressions should be simplified as much as possible.

Question 1 [20 marks]. In this question, $V = \{A \in M_3(\mathbb{K}) \mid A = -A^T\}$, the set of 3×3 antisymmetric matrices regarded as a vector space over a field $\mathbb{K}$. Here A^T is the transpose of A.

(a) What is the dimension of V over $\mathbb{K} = \mathbb{R}$? Justify your answer. [5]

(b) Over $\mathbb{R}$, are the three matrices

$$\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

a basis of V? Justify your answer. [6]

(c) Show that regarded as elements of V over the field $\mathbb{K} = \mathbb{F}_2$ of integers mod 2, the three matrices displayed in part (b) are **not** linearly independent. [5]

(d) Exhibit a basis of V over $\mathbb{F}_2$. You are not required to give a proof, but outline your reasoning. [4]

Question 2 [20 marks].

(a) Let U, W be subspaces of a vector space V. Define what is meant by $V = U \oplus W$. [5]

In the remainder of this question, $V = \mathbb{R}^2$.

(b) Exhibit 1-dimensional subspaces $U, W \subseteq V$ such that $V = U \oplus W$. [5]

(c) Exhibit 1-dimensional subspaces $U, W, X \subseteq V$ such that $V = U + W + X$ and $U \cap W = W \cap X = U \cap X = \{0\}$. [5]

(d) Is it possible to choose the subspaces in part (c) in such a way that $V = U \oplus W \oplus X$? [5]

Justify your answers in parts (b)-(d). You may refer to general results from Lectures.

Question 3 [20 marks].

(a) Let C be the $n \times n$ elementary matrix for the column operation swapping the 1st and 2nd columns of an $m \times n$ matrix. Here $n \geq 2$. Show **using the Leibniz formula** that $\det(C) = -1$. [5]

In the remainder of this question, let v_1, v_2, v_3 be a basis of a vector space V and w_1, w_2 a basis of a vector space W over $\mathbb{R}$. Let $\alpha : V \to W$ be a linear map defined by

$$\alpha(v_1) = 2w_1 + w_2, \quad \alpha(v_2) = w_1 - w_2, \quad \alpha(v_3) = w_1 + w_2.$$

(b) Find $\mathrm{Ker}(\alpha)$ and the **nullity** $\nu(\alpha)$. [7]

(c) Find the **rank** $\rho(\alpha)$ and hence, or otherwise, find $\mathrm{Im}(\alpha)$. Justify your answer. [5]

(d) Is it possible to find new bases for V, W such that the 3×2 matrix corresponding to α with respect to these new bases is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$? Justify your answer. [3]

Question 4 [20 marks]. The **characteristic polynomial** $p_A(x)$ of a general 3×3 matrix A over a field $\mathbb{K}$ necessarily has the form

$$p_A(x) = x^3 - \mathrm{Tr}(A)x^2 + c(A)x - \det(A)$$

for some coefficient $c(A)$ considered as a function of A with values in $\mathbb{K}$.

(a) Show that $c(P^{-1}AP) = c(A)$ for any invertible 3×3 matrix P. [3]

(b) Show that if $c(A) = 0$ and $\det(A) \neq 0$ then A^2 is invertible with inverse $(A - \mathrm{Tr}(A)I_3)/\det(A)$. [6]

(c) Find $p_A(x)$ for

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ -3 & 1 & 0 \end{bmatrix}$$

as a matrix over $\mathbb{R}$. Show your working. [6]

(d) By considering the options for the minimal polynomial $m_A(x)$, or otherwise, show that A in part (c) **cannot** be diagonalised. [5]

You may refer to results from Lectures, including general properties of $p_A(x)$.

Question 5 [20 marks].

(a) Define what it means for two symmetric $n \times n$ matrices A, B to be **congruent**. [**3**]

(b) If B is congruent to A as in part (a), how is the quadratic form associated to B related to the quadratic form associated to A? You are not asked to prove anything. [**3**]

(c) Show that the following quadratic form over $\mathbb{R}$ is **positive definite**:

$$q(x, y, z) = x^2 + 2xy + 3y^2 + z^2.$$ [**7**]

(d) Let V be a real vector space with basis v_1, v_2, v_3 and make V into an inner product space by

$$v_i.v_j = a_{ij}, \quad A = (a_{ij}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find an **orthonormal basis** w_1, w_2, w_3 in terms of v_1, v_2, v_3. Show your working. (Hint: you may wish to consider w_i of the form $w_1 = av_1 + bv_2$, $w_2 = cv_2$ and $w_3 = v_3$ for suitable coefficients a, b, c.) [**7**]

End of Paper.