Queen Mary
University of London

Main Examination period 2017

MTH6140: Linear Algebra II

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: M. Jerrum, M. Walters

Question 1. [20 marks]
In this question, V is a vector space over a field $\mathbb{K}$.

(a) Which of the following operations are valid:

(i) adding a vector to a scalar?

(ii) multiplying a vector by a scalar?

(iii) multiplying two scalars?

(iv) multiplying two vectors? [**4**]

(b) Suppose U is a subset of V. Give easy-to-test conditions for U to be a subspace of V. [**3**]

(c) Prove that the intersection of two subspaces U and W of V is also a subspace of V. [**4**]

(d) Define the **sum** $U+W$ of two subspaces of V. State without proof a relationship between $\dim(U\cap W)$, $\dim(U)$, $\dim(W)$ and $\dim(U+W)$. [**4**]

(e) Let V be $\mathbb{R}^3$ and let U and W be the subspaces

$$U=\left\langle \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\rangle \quad \text{and} \quad W=\left\langle \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\rangle.$$

Determine $\dim(U\cap W)$, $\dim(U)$, $\dim(W)$ and $\dim(U+W)$, briefly justifying your answer. [**5**]

Question 2. [20 marks]
In this question, A, A' and B are $n \times n$ matrices over a field $\mathbb{K}$.

(a) Define the **sign** of a permutation π on $\{1,\dots,n\}$, and write down the Leibniz (sum-over-permutations) formula for the determinant of A. **[6]**

(b) Suppose that A, A' and B agree on all rows except the first. Furthermore, suppose that the first row of B is equal to the sum of the first row of A and the first row of A'. Using the formula from part (a), prove that $\det(B) = \det(A) + \det(A')$. **[6]**

(c) The identity in part (b) of this question is a special case of a property we labelled D1 in the course. State two other properties, D2 and D3 of the determinant function that together with D1 characterise the determinant (i.e., any function from $n \times n$ matrices A to $\mathbb{K}$ satisfying D1–D3 is in fact the determinant of A). **[4]**

(d) Consider the function $\det'$ from $n \times n$ matrices to $\mathbb{K}$ defined as follows: $\det'(A)$ is given by the formula of part (a) but with the summation restricted to **even** permutations, i.e., permutations π with $\mathrm{sign}(\pi) = +1$. One of the properties D1–D3 fails for the modified function $\det'$. Which is it and why? **[4]**

Question 3. [20 marks]

(a) Suppose V is a vector space and $\alpha : V \to V$ is a linear map on V. Define the **kernel** $\mathrm{Ker}(\alpha)$ and **image** $\mathrm{Im}(\alpha)$ of α. **[4]**

(b) Define what it means for a linear map $\pi : V \to V$ to be a **projection** on V. **[3]**

(c) Let π be a projection on V. By considering the identity $v = (v - \pi(v)) + \pi(v)$, prove that $V = \mathrm{Ker}(\pi) + \mathrm{Im}(\pi)$. **[3]**

(d) With π as in part (c), prove that $\mathrm{Ker}(\pi) \cap \mathrm{Im}(\pi) = \{\mathbf{0}\}$. **[3]**

(e) Consider the linear map $I - \pi$ on V where I is the identity map and π is a projection. Prove that $I - \pi$ is a projection. **[3]**

(f) Prove that $\mathrm{Ker}(I - \pi) = \mathrm{Im}(\pi)$ and $\mathrm{Im}(I - \pi) = \mathrm{Ker}(\pi)$. **[4]**

Question 4. [20 marks]
In this question, A is a square matrix with entries in a field $\mathbb{K}$.

(a) Define the **characteristic polynomial** $p_A(x)$ and the **minimal polynomial** $m_A(x)$ of A. **[6]**

(b) State without proof a condition for A to be diagonalisable in terms of the minimal polynomial of A. **[3]**

(c) Compute the characteristic polynomial and minimal polynomials of the real matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix}.$$ **[8]**

(d) Is the matrix A from part (c) diagonalisable? Briefly justify your answer. **[3]**

Question 5. [20 marks]
In this question, α is a linear map on a real inner product space V.

(a) State the condition for a vector $v \in V$ to be an **eigenvector** of α with **eigenvalue** λ. Define the **eigenspace** $E(\lambda, \alpha)$. **[4]**

(b) Explain what it means for subspaces U and W of V to form an **orthogonal decomposition** of V. **[3]**

(c) Define what it means for α to be **self-adjoint**. **[2]**

(d) State a theorem (a version of the Spectral Theorem) about the eigenspaces of a self-adjoint linear map. **[4]**

(e) Consider the matrix

$$A = \begin{bmatrix} 5 & -20 & 22 \\ -20 & 17 & 2 \\ 22 & 2 & -4 \end{bmatrix}.$$

Is A diagonalisable? (This part requires no calculation, but you should justify your answer.) **[2]**

(f) Define the **trace** of a square matrix. **[2]**

(g) Suppose A is as in part (e). Given that two of the eigenvalues of A are 9 and -27, what is the third? (This part requires very little calculation.) **[3]**

End of Paper.