Queen Mary
University of London

Main Examination period 2017

MTH6136 / MTH6136P: Statistical Theory

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: Dr Dudley Stark, Prof Franco Vivaldi

Question 1. [20 marks]

(a) Let T be an estimator of ϕ.

(i) Define what is meant by the **mean square error** and the **bias** of the estimator T. [**2**]

(ii) Prove, directly from the definition of mean square error, that if T is unbiased for ϕ, then the mean square error of T equals the variance of T. [**3**]

(b) Let $Y_1, Y_2, \ldots, Y_n$ be independent Poisson distributed random variables with parameter λ.

(i) Show that $\overline{Y}$ is unbiased for λ. [**3**]

(ii) Show that $\overline{Y}$ is consistent for λ. [**5**]

(iii) Use the central limit theorem to find an approximate $100(1-\alpha)\%$ confidence interval for λ. [**7**]

Question 2. [20 marks] Suppose that $Y_1, Y_2, \ldots, Y_n$ are independent inverse gamma random variables with probability density function

$$f_Y(y) = \frac{\beta^3}{2y^4} e^{-\frac{\beta}{y}}, \quad y > 0,$$

where $\beta > 0$ is a parameter.

(a) Show that the maximum likelihood estimator of β is

$$\frac{3n}{\sum_{i=1}^{n} \frac{1}{Y_i}}.$$

[**6**]

(b) Find the Cramér-Rao lower bound for unbiased estimators of β. Given that $E(Y) = \beta/2$ and $\mathrm{var}(Y) = \beta^2/4$, show that $2\overline{Y}$ is an unbiased estimator of β and determine whether it is an asymptotically efficient estimator. [**9**]

(c) Use Neyman's Factorisation Lemma to show that $\sum_{i=1}^{n} 1/Y_i$ is a sufficient statistic for β. [**5**]

Question 3. [20 marks] Suppose that $Y_1, Y_2, \ldots, Y_n$ are independent lognormal random variables with probability density function

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2 y}} \exp\left\{-\frac{1}{2\sigma^2}(\log y - \mu)^2\right\}, \quad y > 0,$$

where $-\infty < \mu < \infty$ and $\sigma^2 > 0$ are parameters.

(a) Show that this distribution is a member of the exponential family. [**8**]

(b) Write down complete sufficient statistics for μ and σ^2. [**3**]

(c) Given that

$$E(Y) = e^{\mu + \frac{\sigma^2}{2}} \quad \text{and} \quad E(Y^2) = e^{2\mu + 2\sigma^2},$$

find the method of moments estimators for μ and σ^2. Are these estimators minimum variance unbiased estimators? Justify your answer. [**9**]

Question 4. [20 marks] Let $Y_1, Y_2, \ldots, Y_{n_1}$ be exponential random variables with parameter $\lambda_1 > 0$, and let $Y_{n_1+1}, Y_{n_1+2}, \ldots, Y_{n_1+n_2}$ be exponential random variables with parameter $\lambda_2 > 0$, all independent. Let $\overline{Y}_1$ be the mean of the first n_1 observations, and let $\overline{Y}_2$ be the mean of the remaining observations.

(a) Show that the maximum likelihood estimators of λ_1 and λ_2 are $1/\overline{Y}_1$ and $1/\overline{Y}_2$, respectively. [**7**]

(b) Given that $2\lambda_1 n_1 \overline{Y}_1$ and $2\lambda_2 n_2 \overline{Y}_2$ have chi-squared distributions with respective degrees of freedom $2n_1$ and $2n_2$, explain why

$$\left(\sqrt{\frac{\lambda_1}{\lambda_2}}\right)^2 \frac{\overline{Y}_1}{\overline{Y}_2}$$

is a pivot for $\sqrt{\lambda_1/\lambda_2}$. [**7**]

(c) Use this pivot to derive an exact $100(1-\alpha)\%$ confidence interval for $\sqrt{\lambda_1/\lambda_2}$. [**6**]

Question 5. [20 marks] Suppose that Y is a single observation from a population with beta distribution

$$f_Y(y) = \theta y^{\theta-1}, \quad 0 \le y \le 1,$$

where $\theta > 0$ is a parameter.

(a) For testing $H_0 : \theta \le 1$ against $H_1 : \theta > 1$, find the significance level and the power function of the test that rejects H_0 if $Y > 1/2$. [**6**]

(b) Find the most powerful level α test of $H_0 : \theta = 1$ against $H_1 : \theta = 2$. [**8**]

(c) Is there a uniformly most powerful test of $H_0 : \theta = 1$ against $H_1 : \theta > 1$? Justify your answer. [**6**]

End of Paper.