Main Examination period 2017

MTH6132: Relativity

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: S. Beheshti, R. Buzano

Question 1. [15 marks]

(a) In an inertial frame two events occur simultaneously at a distance of 3 metres apart. In a frame moving with respect to this laboratory frame, one event occurs later than the other by 10^{-8} s. By what spatial distance are the two events separated in the moving frame? Supplement your argument with spacetime diagrams. Recall $c = 3 \times 10^8$ m/s. **[8]**

(b) Show that the composition of two Lorentz transformations is itself a Lorentz transformation. You may assume that frames are in standard configuration. **[7]**

Question 2. [10 marks] Let T_{ab} be a general covariant tensor of rank two.

(a) Show that T_{ab} can be expressed as the sum of its symmetric part, $T_{(ab)}$, and anti-symmetric part, $T_{[ab]}$. **[5]**

(b) Prove that $g^{ab}T_{ab} = g^{ab}T_{(ab)}$, where g^{ab} is a general metric tensor. **[5]**

Question 3. [10 marks] A particle of rest mass M moving along the x-axis with speed V decays into two particles, each with a rest mass $\frac{M}{2}$. Assume both particles continue to move along the x-axis. By comparing 4-momenta before and after the event and using conservation of energy, show that the new particles move with the same speed and that the speeds of these particles equal that of the original particle. **[10]**

Question 4. [15 marks] Let A be an arbitrary constant. The metric for a particular two-dimensional spacetime is given by

$$\mathrm{d}s^2 = -e^{2Ar}\mathrm{d}t^2 + \mathrm{d}r^2.$$

(a) Determine the covariant and contravariant components of the metric tensor for this spacetime. **[2]**

(b) Employ the formula for the Christoffel symbols given in the Appendix to calculate the components $\Gamma^a{}_{bc}$ of the connection for this metric. Note the identification $(x^1, x^2) = (t, r)$ is being used here. **[6]**

(c) Use your results from the previous part to compute $R^2{}_{121}$. **[7]**

Question 5. [15 marks]

(a) Use the Lagrangian Method to write down the geodesic equations for the standard Euclidean metric on $\mathbb{R}^2$ in polar coordinates. **[7]**

(b) Prove that straight lines are solutions to these equations. Argue that in fact, all geodesics on the plane must be straight lines. **[8]**

Question 6. [15 marks] The Riemann curvature tensor of a certain 4-dimensional manifold $\mathscr{M}$ is of the form

$$R_{abcd} = K\left(g_{ac}g_{bd} - g_{ad}g_{cb}\right),$$

with K a constant.

(a) Show that $\nabla_e R_{abcd} = 0$, justifying your steps. **[5]**

(b) Prove that the corresponding Ricci tensor is proportional to the metric, making $\mathscr{M}$ an Einstein manifold. **[5]**

(c) Prove that $\mathscr{M}$ has constant scalar curvature (a fact which is true for all Einstein manifolds of dimension $n \geq 3$). In the case of Einstein manifolds, we necessarily have $R_{ab} = \frac{R}{n} g_{ab}$. You may assume this equality. **[5]**

Question 7. [10 marks] Recall that X^a is called a Killing vector if it satisfies the following equation

$$\nabla_{(a}X_{b)} = \nabla_a X_b + \nabla_b X_a = 0.$$

Prove that if T_{ab} is a symmetric tensor and X^a is a Killing vector, then the vector $V_a = T_{ab}X^b$ satisfies

$$\nabla^a V_a = 0$$

whenever $\nabla^a T_{ab} = 0$. **[10]**

Question 8. [10 marks] Consider a metric perturbation of Minkowski spacetime, given by

$$g_{ab} = \eta_{ab} + \varepsilon h_{ab},$$

where ε is a small constant, $|\varepsilon| \ll 1$. Show that to first order, $g^{ab} = \eta^{ab} - \varepsilon h^{ab}$.
Hint: Given $g_{ab}g^{bc} = \delta_a^c$, assume $g^{ab} = \eta^{ab} + \varepsilon\alpha h^{ab}$ for some constant α to be determined. Expand $g_{ab}g^{bc}$ and use the symmetry of η to deduce $\alpha = -1$. **[10]**

End of Paper—An appendix of 1 page follows.

You are reminded of the following information, which you may use without proof.

- Lower case Latin indices run from 0 to 3.
- The metric tensor of the Minkowski spacetime is η_{ab} such that
$$ds^2 = \eta_{ab}dx^a dx^b = -c^2dt^2 + dx^2 + dy^2 + dz^2$$
- The Lorentz transformations between two frames F and F' in standard configuration are given by
$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad y' = y, \quad z' = z$$
where
$$\gamma = \frac{1}{\sqrt{1-(v^2/c^2)}}$$
and F' is moving with speed v relative to F.
- The covariant derivative of a covariant vector is given by
$$\nabla_a V_b = \partial_a V_b - \Gamma^f{}_{ba} V_f.$$
- The covariant derivative of a contravariant vector is given by
$$\nabla_a V^b = \partial_a V^b + \Gamma^b{}_{af} V^f.$$
- The metric tensor satisfies:
$$g_{ab}g^{bc} = \delta_a{}^c.$$
- Christoffel symbols (connection):
$$\Gamma^m{}_{ij} = \frac{1}{2}g^{mk}(\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij}).$$
- The Riemann curvature tensor:
$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}.$$
- Euler–Lagrange equations:
$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0$$
- Geodesic equations:
$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0.$$

End of Appendix.