MTH6132: Relativity

Duration: 2 hours

Date and time: 5th May 2016, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): S. Beheshti

Question 1. Let F and F' denote two inertial reference systems moving with velocity v with respect to each other. In F, two events occur simultaneously at $t = 0$, separated by a distance X along the x-axis. The time interval between the events in F' is T.

(a) Draw a 2-dimensional spacetime diagram describing the situation, including both F and F'. You may assume units for which $c = 1$. **[4]**

(b) Show that the spatial distance between the two events in F' is $\sqrt{X^2 + T^2}$. **[4]**

(c) Determine the relative velocity v of the frames F, F' in terms of X and T. You may assume $c = 1$ in your calculations. **[7]**

Question 2. Let $\bar{A}$ and $\bar{B}$ denote two arbitrary 4-vectors in Minkowski spacetime.

(a) Define what is meant by the scalar product $\bar{A} \cdot \bar{B}$. What does it mean to say $|\bar{A}|^2$ is an invariant? **[4]**

(b) Using the fact that $|\bar{A}|^2$, $|\bar{B}|^2$ and $|\bar{A} + \bar{B}|^2$ are invariants, show that the scalar product $\bar{A} \cdot \bar{B}$ is also an invariant. **[4]**

(c) Show that the sum of any two orthogonal spacelike vectors is also spacelike. **[7]**

Question 3. The metric for a particular two-dimensional spacetime is given by

$$\mathrm{d}s^2 = \frac{1}{y^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)$$

(a) Determine the covariant and contravariant components of the metric tensor for this spacetime. **[2]**

(b) Employ the formula for the Christoffel symbols given in the Appendix to calculate the components $\Gamma^2{}_{11}$, $\Gamma^1{}_{12}$ and $\Gamma^2{}_{22}$ of the connection for this metric. [Note the identification $(x^1, x^2) = (x, y)$ is used here.] **[6]**

(c) Assuming that the components in part (b) are the only nonzero ones, confirm that the $R^2{}_{121}$ component of the Riemann tensor for this metric is $-\frac{1}{y^2}$. Given that Gauss curvature is given by $K = R_{1212}/(g_{11}g_{22} - g_{12}g_{21})$, can this metric describe a flat spacetime? **[7]**

Question 4.

(a) Assume ∇ is the Levi-Civita connection of a metric g_{ab}. Using properties of this covariant derivative, simplify fully the expression

$$\nabla_a \left(g_{bc} S^{bc} \right) .$$

[3]

(b) Let X^a be the tangent vector to a geodesic given by $x^a(\lambda)$. Using part (a), show that the norm of this tangent vector is conserved along geodesics, i.e.,

$$X^a \nabla_a \left(|X|^2 \right) = 0.$$

[7]

Question 5. In this question consider units for which $c = 1$. A particle has rest mass m_0. Whilst at rest, it emits a photon and, as a result, its rest mass is reduced to $m_0/2$. By comparing components of the 4-momenta before and after the event, show that the speed of the particle after the reduction of mass is $3/5$. Show also that the energy $E = h\nu$ of the photon is $3m_0/8$. **[10]**

Question 6.

(a) It can be shown that in a Local Inertial Frame the Riemann tensor can be expressed in the form

$$R_{abcd} = \frac{1}{2} \left(\partial_d \partial_a g_{bc} + \partial_c \partial_b g_{ad} - \partial_c \partial_a g_{bd} - \partial_d \partial_b g_{ac} \right)$$

at a specific point p. Explain what is meant by a Local Inertial Frame and employ the expression above to show that

$$R_{abcd} = -R_{bacd}$$

at p. Is this relation valid in an arbitrary frame of reference? Explain your reasoning. **[5]**

(b) Suppose that the curvature of a spacetime satisfies the equation

$$R_{ab} - \frac{1}{2} R g_{ab} + \lambda g_{ab} = 0,$$

where λ is a constant. Define the Ricci scalar and show that it satisfies

$$R = 4\lambda.$$

Hint: You will also need to prove and use the fact that $g_{ab} g^{ab} = 4$. **[5]**

Question 7.

(a) Write down the transformation laws under general coordinate transformations for a $(1,0)$-tensor and a $(0,2)$-tensor, respectively. Use this to show that the product of these tensors is a tensor of type $(1,2)$. [**5**]

(b) Prove that if $W_{ab}{}^{c}$ is a $(1,2)$-tensor, then $W_{ab}{}^{b}$ is a $(0,1)$-tensor. [**5**]

Question 8. Consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right).$$

(a) What physical situation is described by this metric? What happens if $M = 0$? [**2**]

(b) At what two values of r is this metric in the above form singular? Re-express the Schwarzschild metric in terms of *Eddington-Finkelstein coordinates*, given by $(\hat{t}, r, \theta, \varphi)$, where

$$\hat{t} = t + 2GM \ln |r - 2GM|.$$

Show that in these coordinates, one of the two singularities is removed.

[**6**]

(c) Use the Euler-Lagrange equations to derive the geodesic equations obeyed by a photon for this metric. [**7**]

End of Paper—An appendix of 1 page follows.

You are reminded of the following information, which you may use without proof.

- Lower case Latin indices run from 0 to 3.
- The metric tensor of the Minkowski spacetime is η_{ab} such that

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

- The Lorentz transformations between two frames F and F' in standard configuration are given by

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad y' = y, \quad z' = z$$

  where

$$\gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}}$$

  and F' is moving with speed v relative to F.

- The covariant derivative of a covariant vector is given by

$$\nabla_a V_b = \partial_a V_b - \Gamma^f{}_{ba} V_f.$$

- The covariant derivative of a contravariant vector is given by

$$\nabla_a V^b = \partial_a V^b + \Gamma^b{}_{af} V^f.$$

- The metric tensor satisfies:

$$g_{ab} g^{bc} = \delta_a{}^c.$$

- Christoffel symbols (connection):

$$\Gamma^m{}_{ij} = \frac{1}{2} g^{mk}(\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij}).$$

- The Riemann curvature tensor:

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}.$$

- Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0$$

- Geodesic equations:

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0.$$

End of Appendix.