Queen Mary
University of London

Main Examination period 2020 – May/June – Semester B
Online Alternative Assessments

MTH6127 / MTH6127P: Metric Spaces and Topology

This paper has two sections.
You should attempt all the questions in Section A. In Section B you may attempt as many questions as you wish. Except for the award of a bare pass, only the best TWO questions answered in Section B will be counted.

In completing this assessment, you may use books, notes, and the internet. You may use calculators and computers, but you should show your working for any calculations you do. You must not seek or obtain help from anyone else.

At the start of your work, please **copy out and sign** the following declaration:

> I declare that my submission is entirely my own, and I have not sought or obtained help from anyone else.

All work should be **handwritten**, and should **include your student number**.

You have **24 hours** in which to complete and submit this assessment. When you have finished your work:

- scan your work, convert it to a **single PDF file** and upload this using the upload tool on the QMplus page for the module;
- e-mail a copy to maths@qmul.ac.uk with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are not expected to spend a long time working on this assessment. We expect you to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the assessment period, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: B. Noohi, R. Buzano

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In this examination $\mathbb{R}$ stands for the set of real numbers and $\mathbb{N} := \{1, 2, 3, \ldots\}$ stands for the set of natural numbers.

Section A

You should attempt both questions in this section.

Question 1 [20 marks]. Let (X, d) be a metric space.

(a) Are the following definitions correct? If a definition is incorrect, explain why it is not equivalent to the correct definition and give a corrected version of it: [**5**]

> A subset $\Omega \subseteq X$ is **open** if for each point $x \in \Omega$ there exists an $r > 0$ such that $B_r(x) \subseteq \Omega$, where $B_r(x)$ stands for the open ball of radius r around x.
>
> A subset $F \subseteq X$ is **closed** if for each point $x \in F$ there exists an $r > 0$ such that $\overline{B}_r(x) \subseteq F$, where $\overline{B}_r(x)$ stands for the closed ball of radius r around x.

(b) Prove, starting from the definitions, that if U, V, and W are open sets in X, then $U \cap V \cap W$ is also an open set in X. [**5**]

(c) Is it possible for an open ball $B_r(x)$ of a metric space to be a closed set? If your answer is no, prove your claim. If your answer is yes, give an example. [**4**]

(d) Is the following definition correct? If the definition is incorrect, explain why it is not equivalent to the correct definition and give a corrected version of it: [**2**]

> Two metrics d and d' on the same set X are called **equivalent** if every open ball in the metric d is an open ball in the metric d' and vice versa.

(e) Consider the metric on $\mathbb{R}^n$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Prove that this metric is not equivalent to the Euclidean metric on $\mathbb{R}^n$. [**4**]

Question 2 [20 marks]. Let (X, τ) be a topological space.

(a) Let $Y \subseteq X$ be a subset. Is the following definition correct? If the definition is incorrect, explain why it is not equivalent to the correct definition and give a corrected version of it: [**3**]

> The **relative topology** τ_Y on Y is the topology associated to the metric induced from a metric on X.

(b) Is the following definition correct? If the definition is incorrect, explain why it is not equivalent to the correct definition and give a corrected version of it: [**3**]

> A topological space (X, τ) is called **Hausdorff** if there exist disjoint open sets Ω_1 and Ω_2 such that for every $x \neq y$ in X, x and y are not both in one of Ω_1 or Ω_2.

(c) If (X, τ) is Hausdorff and $Y \subseteq X$, prove that the relative topology τ_Y on Y is also Hausdorff. [**4**]

(d) Give an example of a topology on the set $X = \{a, b, c\}$, different from the trivial topology, that is not Hausdorff. Briefly justify your answer. [**3**]

(e) Prove that the only Hausdorff topology on a finite set X is the discrete topology. [**7**]

Section B

You may attempt as many questions as you wish in this section. Except for the award of a bare pass, only the best TWO questions answered in this section will be counted.

Question 3 [30 marks].

(a) Is the following definition correct? If the definition is incorrect, explain why it is not equivalent to the correct definition and give a corrected version of it: [**4**]

> Let $A \subseteq X$ be a subset of a topological space X. The **interior** $int(A)$ of A is an open subset of A that contains any other open subset of A.

(b) Let $X = \mathbb{R}$ be endowed with the standard topology induced by the metric $d(x, y) = |x - y|$, and let $A := (0, 1) \cup ((1, 2) \cap \mathbb{Q}) \cup \{3\} \cup [4, \infty)$. Find the following sets [**12**]

$$B := int(A), \qquad C := cl(int(A)), \qquad D := int(cl(int(A))),$$
$$E := cl(A), \qquad F := int(cl(A)), \qquad G := cl(int(cl(A))).$$

(c) Let (X, τ_X) and (Y, τ_Y) be topological spaces. Is the following definition correct? If the definition is incorrect, explain why it is not equivalent to the correct definition and give a corrected version of it: [**4**]

> The function $f : X \to Y$ is **continuous**, if for every $a \in X$,
>
> $$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X, d_X(x, a) < \delta \; : \; d_Y(f(x), f(a)) < \varepsilon.$$

(d) Let $f : X \to Y$ be a continuous map. Suppose that X is connected. Prove that $f(X)$, the image of f in Y, is a connected subset of Y. [**5**]

(e) Prove that S^1 is not homeomorphic to an open interval in $\mathbb{R}$.
(Hint: what happens if you remove a point from either of these spaces?) [**5**]

Question 4 [30 marks].

(a) Is the following definition correct? If the definition is incorrect, explain why it is not equivalent to the correct definition and give a corrected version of it: [**4**]

> A topological space (X, τ) is **metrisable** if there exists a metric d on X such that for every $\Omega \in \tau$, Ω can be written as a union of open balls of the metric d.

(b) Let X be an infinite set, and consider the *finite-complement topology* on X defined by

$$\tau = \{\Omega \subseteq X \ : \ \Omega = \emptyset \text{ or } \Omega^c \text{ is finite}\}.$$

Prove that X with this topology is not metrisable. (You do not need to prove that τ is a topology.) [**8**]

(c) Let (X, d) be a metric space. Let $A \subseteq X$ be a subset and let x be a point in the closure of A. Prove that there exists a sequence $(x_n)_{n=1}^{\infty}$ converging to x such that $x_n \in A$ for all n. Can we always find such a sequence in which the x_n are pairwise distinct? [**8**]

(d) Is the following definition correct? If the definition is incorrect, explain why it is not equivalent to the correct definition and give a corrected version of it: [**3**]

> A topological space X is **compact** if it has an open cover $\{\Omega_i\}_{i \in I}$, where I is a finite set.

(e) Using only the definition of compactness, prove that the subset $A = \{\frac{1}{n} \ : \ n \in \mathbb{N}\} \subset \mathbb{R}$ is not compact, but the subset $B = \{0\} \cup \{\frac{1}{n} \ : \ n \in \mathbb{N}\} \subset \mathbb{R}$ is compact. [**7**]

Question 5 [30 marks].

(a) Define what is meant by a **Cauchy sequence** in a metric space (X, d). Is the following definition of **completeness** correct? If the definition is incorrect, explain why it is not equivalent to the correct definition and give a corrected version of it: **[6]**

> A metric space (X, d) is **complete** if every convergent sequence in X is a Cauchy sequence.

(b) Let X be a complete metric space. Let $A \subseteq X$ be a closed subset of X. Is it true that every Cauchy sequence in A converges to a point in A? Prove your claim. **[6]**

(c) Let $X = \mathcal{B}(0, 1)$ denote the space of bounded functions from the open interval $(0, 1)$ to $\mathbb{R}$ with the sup-metric

$$d(f, g) := \sup_{x \in (0,1)} |f(x) - g(x)|, \qquad \forall f, g \in \mathcal{B}(0, 1).$$

Prove that the sequence $(f_n)_{n=1}^{\infty}$, where $f_n(x) = \frac{1}{2+n\sqrt{x}}$, does not converge in X. (Hint: first determine the pointwise limit of the sequence.) **[8]**

(d) Let $f : X \to X$ be a function on a metric space (X, d) such that $d(f(x), f(y)) < d(x, y)$ for all distinct points $x, y \in X$. Is it true that f is a contraction mapping? Justify your answer.

Suppose we know that f has a fixed point. Is it true that the fixed point is unique? Justify your answers. Justify your answer. **[5]**

(e) Consider $\mathbb{R}^3$ with the Euclidean metric. Prove that the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x_1, x_2, x_3) = \left(\frac{1}{3}x_1 + 1,\ \frac{1}{3}x_2 + 2,\ \frac{1}{3}x_3\right)$$

is a contraction mapping. What is the fixed point of f? **[5]**

End of Paper.