Queen Mary
University of London

Main Examination period 2018

MTH6127: Metric Spaces and Topology

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

This paper has two sections.
You should attempt all the questions in Section A. In Section B you may attempt as many questions as you wish. Except for the award of a bare pass, only the best TWO questions answered in Section B will be counted.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: R. Buzano, S. Beheshti

In this examination $\mathbb{R}$ stands for the set of real numbers and $\mathbb{Q}$ stands for the set of rational numbers.

Section A

Question 1. [10 marks] Let V be a real vector space.

(a) Define what is meant by a **norm** on V. **[2]**

(b) Define what is meant by a **metric** on V. **[2]**

(c) Prove that every norm on V induces a metric on V. **[6]**

Question 2. [10 marks] Let (X, d) be a metric space.

(a) Define what is meant by an **open ball** $B_r(x)$ in (X, d). **[2]**

(b) Let $A \subseteq X$ be a subset. Define what it means for A to be an **open set**. **[2]**

(c) Prove that an open ball $B_r(x)$ is indeed an open set. **[6]**

Question 3. [10 marks] Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : X \to Y$ be a function.

(a) Define what it means for the function $f : X \to Y$ to be **continuous**. **[2]**

(b) Define what it means for X to be **connected**. **[2]**

(c) Prove that if $f : X \to Y$ is continuous and surjective and X is connected, then Y is connected. **[6]**

Question 4. [10 marks] In a metric space (X, d), we say the sequence $(x_n)_{n=1}^{\infty}$ converges to $x \in X$ if and only if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N \; : \; x_n \in B_\varepsilon(x). \tag{1}$$

(a) Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ in a topological space (X, τ) to **converge** to $x \in X$. **[2]**

(b) Prove that if the topology τ is induced by a metric d, then the definition you wrote down in part (a) is equivalent to the definition given in (1). **[8]**

Section B

Question 5. [30 marks] Let X be an infinite set.

(a) Define what is meant by a **topology** on X. **[2]**

(b) Prove that the collection of sets $\tau_1 := \{A \subseteq X \ : \ A = \emptyset \text{ or } A^c \text{ is a finite set}\}$ defines a topology on X. Here $A^c = X \setminus A$ is the complement of A. **[6]**

(c) Prove that the collection of sets $\tau_2 := \{A \subseteq X \ : \ A = X \text{ or } A \text{ is a finite set}\}$ does **not** define a topology on X. **[6]**

(d) Define what it means for a topology τ on X to be **Hausdorff**. **[2]**

(e) Prove that the topology τ_1 given in part (b) is **not** Hausdorff. **[6]**

(f) Prove that in every Hausdorff topological space (X, τ), any set containing exactly one point is closed. **[8]**

Question 6. [30 marks]

(a) Let (X, τ) be a topological space and $A \subseteq X$ a subset. Define what is meant by the **interior** of A (denoted $int(A)$) and by the **closure** of A (denoted $cl(A)$). **[2]**

(b) Let $X = \mathbb{R}$ be endowed with the standard topology induced by the metric $d(x, y) = |x - y|$ and let $A := (0, 1) \cup (1, 2) \cup \{3\} \cup ([4, \infty) \cap \mathbb{Q})$. Without justification, find the following sets

$$B := int(A), \qquad C := cl(int(A)), \qquad D := int(cl(int(A))),$$
$$E := cl(A), \qquad F := int(cl(A)), \qquad G := cl(int(cl(A))).$$

(*Hint: All of the seven sets A, B, C, D, E, F, and G are different.*) **[12]**

(c) Define what is meant by a **compact** subset of a topological space (X, τ). **[2]**

(d) Exactly one of the seven sets A, B, C, D, E, F, and G from part (b) is compact. Which one? Justify your answer. (You may use any result from the lectures provided you make it clear what you are using.) **[6]**

(e) Prove that the union of finitely many compact sets in a topological space (X, τ) is also compact. **[8]**

Question 7. [**30 marks**]

(a) Define what is meant by a **Cauchy sequence** in a metric space (X, d). [**2**]

(b) Let $Z = \mathbb{R}$ and $d_Z(x, y) := |e^{-x} - e^{-y}|$. Prove that the sequence $(x_n)_{n=1}^{\infty}$ given by $x_n = n$ is a Cauchy sequence in (Z, d_Z). (You do not need to prove that d_Z is indeed a metric on Z.) [**6**]

(c) Is the space (Z, d_Z) given in part (b) complete? Justify your answer. [**6**]

(d) Given a metric space (X, d), when do we say that a map $f : X \to X$ is a **contraction mapping**? [**2**]

(e) Prove that every contraction mapping $f : X \to X$ is continuous. [**6**]

(f) Let (X, d) be a complete metric space and $f : X \to X$ a map for which there exists $k \in \mathbb{N}$ such that f^k is a contraction mapping. Prove that f has a unique fixed point. (You are allowed to use Banach's Fixed Point Theorem without proof.) [**8**]

End of Paper.