Queen Mary University of London

B. Sc. Examination by course unit 2015

MTH6127: Metric Spaces and Topology

Duration: 2 hours

Date and time: 29th May 2015, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiner(s): M. Farber

In this examination $\mathbb{R}$ stands for the set of real numbers.

Section A: Each question carries 10 marks. You should attempt all four questions.

Question 1.

(a) Give the definition of *a metric space* (X, d). **[2]**

(b) Define what is meant by *an open ball* $B(c, r)$ in a metric space (X, d). **[2]**

(c) Explain what it means for a set $U \subseteq X$ to be *open*. **[2]**

(d) Prove that the union of any family of open sets is an open set. **[4]**

Question 2.

(a) When do we say that a sequence $\{x_n\}_{n\geqslant 1}$ of points in a metric space X *converges*? **[2]**

(b) Give the definition of *a Cauchy sequence* in a metric space (X, d). **[2]**

(c) Prove that any convergent sequence is a Cauchy sequence. **[2]**

(d) Explain what is meant for a metric space (X, d) to be *complete*. **[2]**

(e) Give an example of a metric space which is not complete. **[2]**

Question 3.

(a) Prove that a closed subset of a complete metric space is complete with respect to the induced metric. **[3]**

(b) Let X be a metric space and let $A \subseteq X$ be a subset which is not closed. Show that A is not complete with respect to the induced metric. **[3]**

(c) Which of the following subsets of $\mathbb{R}$ are complete when considered as subspaces of $\mathbb{R}$ with the usual metric? Briefly explain your answer.

(i) $\{n^{-2};\, n = 1, 2, \ldots\}$, **[2]**

(ii) $\{n^{-2}; n = 1, 2, \ldots\} \cup \{0\}$. **[2]**

Question 4.

(a) Define the sup metric on the set $C[0, \pi]$ of all real continuous function on the closed interval $[0, \pi]$. **[3]**

(b) Is $C[0, \pi]$ complete? (No proof is required.) **[3]**

(c) Decide whether the sequence of functions

$$f_n(x) = \sin(nx), \quad x \in [0, \pi],$$

converges in $C[0, \pi]$ with respect to the sup metric. **[4]**

Section B: Each question carries 30 marks. You may attempt all three questions. Except for the award of a bare pass, only marks for the best two questions will be counted.

Question 5.

(a) Define what is meant by the closed ball $B[c, r] \subseteq X$ in a metric space X. **[2]**

(b) Show that, viewed as subsets of X, the open ball is open and the closed ball is closed. **[5]**

(c) Give the $\varepsilon - \delta$ definition of continuity of a map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y). **[3]**

(d) Show that if a map $f : X \to Y$ is continuous then for any open set $U \subseteq Y$ the preimage $f^{-1}(U) \subseteq X$ is open. **[4]**

(e) Give an example of a non-constant continuous map $f : \mathbb{R} \to \mathbb{R}$ and an open subset $U \subseteq \mathbb{R}$ such that the image $f(U) \subseteq \mathbb{R}$ is not open. **[5]**

(f) Show that if a map $f : X \to Y$ is continuous then for any closed set $F \subseteq Y$ the preimage $f^{-1}(F) \subseteq X$ is closed. **[4]**

(g) Is it true that the image of a closed set under a continuous map is closed? Explain your answer. **[7]**

Question 6.

(a) When do we say that a metric space is *compact*? [**2**]

(b) Prove that any compact subset of a metric space is bounded. [**5**]

(c) Prove that any compact subset of a metric space is closed. [**5**]

(d) State the criterion of compactness for subsets of the Euclidean space $\mathbb{R}^n$. [**3**]

(e) Which of the following subsets of the real line $\mathbb{R}$ are compact; briefly explain your answer:

(i) $[0, 1]$; [**3**]

(ii) $(0, 1)$; [**3**]

(iii) $[0, \infty)$; [**3**]

(iv) $\mathbb{R}$; [**3**]

(v) $\{n^{-1}; n = 1, 2, \dots\}$. [**3**]

Question 7.

(a) Let (X, d) be a metric space. When do we say that a mapping $f : X \to X$ is *a contraction*? [**4**]

(b) State the contraction mapping theorem. [**5**]

(c) Consider $\mathbb{R}^2$ with d_1 metric, i.e. $d_1(v, v') = |x-x'|+|y-y'|$ where $v = (x, y)$ and $v' = (x', y')$. Is this metric space complete? [**5**]

(d) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(v) = (\frac{1}{2}y, \frac{1}{2}(x + 1))$ where $v = (x, y)$. Show that f is a contraction with respect to d_1-metric. [**10**]

(e) Find the fixed point of f. [**6**]

End of Paper.