MTH6120 / MTH6120P: Further Topics in Mathematical Finance

Duration: 2 hours

Date and time: 13th May 2016, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators must not be used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): Dr S. del Bano Rollin

In this exam we will assume that assets pay no dividends.
Please write clearly. Text that is unreadable cannot be graded.

Question 1 (25 marks).

(a) What is the **return** on an investment? What is the **annualised return**? Calculate these amounts in the case I invest $\pounds 100$ which after two years results in $\pounds 105$. **[5]**

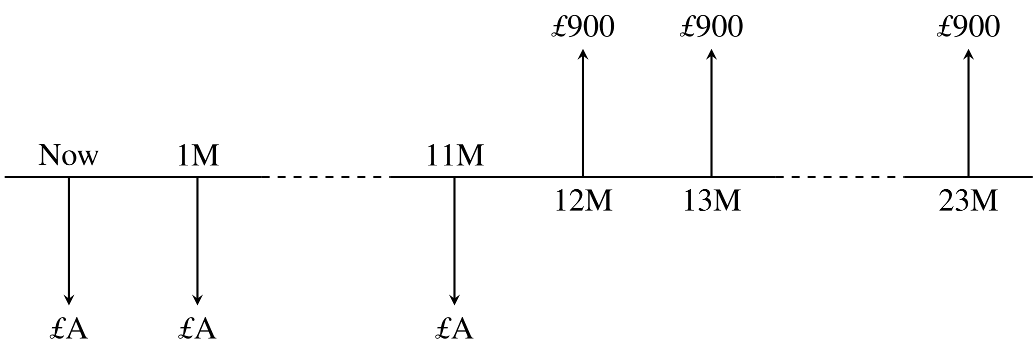
(b) Assume that bank A offers a monthly compounded interest rate of r_{1M} and bank B offers a yearly compounded interest rate of r_{1Y}. Show that if $1 + r_{1Y} < (1 + r_{1M}/12)^{12}$ there is an arbitrage opportunity. Explain how you would take advantage of this arbitrage. **[5]**

(c) Let r be the quarterly compounded interest rate and r_{eff} the corresponding effective rate. Is r_{eff} bigger or smaller than r? Prove your claim.

Hint: *Quarterly refers to a period of three months. Assume* $r > 0$. **[4]**

(d) You wish to fund a gap year starting the following year. You will need $\pounds 900$ every month for 12 months paid at the start of the month and starting in 12 months from now. Given a monthly compounded rate of r, how much money should you save every month starting today in order to fund this? You may use the variable $\alpha = 1/(1 + r/12)^{12}$. Simplify the result as much as possible.

You can use the following diagram to help understand the sequence of cashflows:



[7]

(e) In the presence of deterministic variable interest rates, if $P(t)$ indicates the value of an account that starts with $P(0) = \pounds 1$, how can you derive the instantaneous interest rate $r(t)$? **[4]**

Question 2 (25 marks).

(a) State the Arbitrage Theorem. Use it to prove that $\mathbb{E}(S(T)) = \mathrm{e}^{rT} S(0)$ and Call $= \mathrm{e}^{-rT} \mathbb{E}\left((S(T) - K)^+\right)$. The notation is as in the lectures: $S(t)$ is the price of the asset at time t, r is the continuously compounded rate (assumed constant), and Call indicates the price of a European call option with strike K and expiring at time T. **[5]**

(b) Using the expression for the value of a call option in the previous question and the equivalent equality for a put option:

$$\text{Put} = \mathrm{e}^{-rT} \mathbb{E}\left((K - S(T))^+\right),$$

derive the formula for Call-Put parity, Call $-$ Put $= \mathrm{e}^{-rT}(F_T - K)$, where $F_T = \mathrm{e}^{rT} S(0)$ is the forward price. Indicate how call-put parity might be useful. **[5]**

(c) From the Black-Scholes expression for the price of a call option,

$$\text{Call} = \mathrm{e}^{-rT}\left(N(d_1) F_T - N(d_2) K\right),$$

derive using Call-Put parity the Black-Scholes price of a put option. **[5]**

(d) In the lectures we have proved that the price of a call option is convex in the variable K. What can you say, by using Call-Put parity, about the convexity of the price of a put option?

Hint: *Show that ∂^2Call$/\partial K^2 = \partial^2Put/\partial K^2$.* **[5]**

(e) Using the equality $F_T n(d_1) = K n(d_2)$ proved in the lectures, derive an expression for the vega of a call option defined as $\mathcal{V} = \partial\text{Call}/\partial\sigma$.

Hint: *You might wish to use the formulæ for d_1 and d_2 in the appendix.* **[5]**

Question 3 (25 marks). A straddle with strike K and expiry T is a portfolio consisting of:

- a call option to buy one unit of the asset with strike K and expiring at time T, and
- a put option to sell one unit of the asset with strike K and expiring at time T.

So that at time T you have both the option to buy and the option to sell the asset at price K.

(a) Write down the payoff function for a straddle as describe above. Draw a graph of the payoff function. **[5]**

(b) Show that the price of a straddle is

$$\mathrm{e}^{-rT}\left[F_T\left(2N(d_1)-1\right)-K\left(2N(d_2)-1\right)\right]$$

Hint: *You might wish to use the equality* $N(-x)=1-N(x)$. **[5]**

(c) Show that the delta of a straddle is $2N(d_1)-1$. **[5]**

(d) Find the strike, K, such that the delta of the straddle is zero.

Hint: *The increasing function* $N(x)$ *reaches* $1/2$ *exactly at* $x=0$. **[5]**

(e) The price of a call option is decreasing as a function of strike K. Explain why this is the case using a financial intuitive argument or example; and also prove it mathematically by using the payoff functions for a call option with strike K and a call option with a larger strike $K+\varepsilon$ ($\varepsilon>0$). **[5]**

Question 4 (25 marks).

(a) Explain the meaning of systemic risk and idiosyncratic risk in the CAPM. [**5**]

(b) Prove the equality $\mathbb{V}ar\,(R_i) = \beta_i^2 \mathbb{V}ar\,(R_M) + \mathbb{V}ar\,(e_i)$, where the variables are as in the CAPM. Namely: R_i indicates the return of investment i over the investment period, R_M indicates the return of the market, e_i is the idiosyncratic part of the return, and β_i is the beta of the asset. [**5**]

(c) Consider the function $u(x) = x^\beta$ defined for $x > 0$. For what values of β is $u(x)$ a risk-averse utility function. [**5**]

(d) Consider investing a fraction α of your capital $£X$ in an investment that doubles the invested amount with probability p and halves it with probability $q = 1 - p$. Write down the payoff function. Calculate the expected payoff. Simplify the result as much as possible. [**5**]

As you know from the lectures, the expected payoff calculated in the previous question is not always the best form to evaluate an investment. A better way is to evaluate the expected utility of the payoff.

(e) Assume that your utility function is $u(x) = \sqrt{x}$. Find the α that maximizes the expected utility of the payoff in the previous question. [**5**]

End of Paper—An appendix of 1 page follows.

The Black-Scholes Formula

The price of a **European call option** with strike K and years to expiry T is

$$\begin{aligned}
\text{Call} &= \mathrm{e}^{-rT}\left(F_T N(d_1) - K N(d_2)\right) \\
d_1 &= \frac{\log(F_T/K)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \\
d_2 &= \frac{\log(F_T/K)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}
\end{aligned}$$

where $F_T = \mathrm{e}^{rT} S$ is the forward price, r the interest rate, $S = S(0)$ the current value of the asset, and σ its volatility.
$N(x)$ is the standard normal cumulative distribution function. We also use $n(x) = N'(x)$ to denote the density of the standard normal distribution, $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$.
The price of a **European put option** is

$$\text{Put} = \mathrm{e}^{-rT}\left(-F_T N(-d_1) + K N(-d_2)\right).$$

The delta of a call option is

$$N(d_1),$$

and the delta if a put option is

$$-N(-d_1).$$

Geometric sum

For integers $a \leq b$, we have:

$$x^a + x^{a+1} + \cdots + x^b = \frac{x^a - x^{b+1}}{1 - x}.$$

End of Appendix.