Queen Mary
University of London

Main Examination period 2022 – January – Semester A

MTH6107: Chaos & Fractals

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: O. Jenkinson, M. Shamis

Throughout this exam, the notation $d_1d_2d_3d_4d_5d_6d_7d_8d_9$ will denote your 9-digit QMUL student ID number, where d_i is an element of $\{0,1,2,3,4,5,6,7,8,9\}$ for $1 \le i \le 9$. Associated to this, the number x_0 is defined to be the element of the unit interval whose decimal expansion is $x_0 = 0.d_1d_2d_3d_4d_5d_6d_7d_8d_9$, in other words $x_0 = \sum_{i=1}^{9} d_i/10^i$. The distinct digits in your student ID number, written in increasing order, will be denoted $a_1 < a_2 < a_3 < \ldots < a_k$, where $k \le 9$ is the cardinality of the set $\{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9\}$.

Question 1 [25 marks]. For a_4 and a_5 your 4th and 5th smallest student ID digits respectively, let $c = a_5/a_4$ and suppose the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 2cx + c^2 + c$.

(a) Write down c and $x_0 = 0.d_1d_2d_3d_4d_5d_6d_7d_8d_9$, and compute the first three iterates $f(x_0)$, $f^2(x_0)$, $f^3(x_0)$, accurate to at least 4 decimal places. **[3]**

(b) Draw the graph of f, and determine all of its fixed points. **[6]**

(c) Determine an eventually fixed point of f that is not a fixed point. **[4]**

(d) Determine which of the fixed points of f are attracting, and which are repelling, taking care to justify your answer. **[6]**

(e) If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x^2 - 2c^{-1}x + c^{-2} + c^{-1}$, determine whether there is a topological conjugacy from f to g, taking care to justify your answer. **[6]**

Question 2 [25 marks]. Let $x_0 = 0.d_1d_2d_3d_4d_5d_6d_7d_8d_9$, and let $a_1 < a_2 < a_3 < \ldots$ be your distinct student ID digits written in ascending order.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x + e^{a_4 x} = x + \exp(a_4 x)$, determine all fixed points of f, being careful to justify your answer. **[5]**

(b) Write down a_3 and a_4, and give an explicit example of a diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $\{a_3, a_4\}$ is a 2-cycle. **[5]**

(c) Write down x_0, and give an explicit example of a diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that x_0 is an attracting fixed point, being careful to justify your answer. **[5]**

(d) Write down your student ID digit d_9, and give explicit examples of diffeomorphisms $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that d_9 is an attracting fixed point for f, a repelling fixed point for g, and an attracting fixed point for $f \circ g$. **[5]**

(e) Give an explicit example of a function $f : \mathbb{R} \to \mathbb{R}$ such that 0 is an attracting fixed point, and its basin of attraction is precisely the open interval $(-a_4, a_4)$, being careful to justify your answer. **[5]**

Question 3 [25 marks].

(a) Write down, in order, the smallest 5 digits $a_1 < a_2 < a_3 < a_4 < a_5$ in your student ID number. [**1**]

(b) Sketch the graph of the continuous piecewise-linear function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(a_1) = a_3$, $f(a_2) = a_5$, $f(a_3) = a_4$, $f(a_4) = a_2$, $f(a_5) = a_1$, is linear on each of the intervals $[a_1, a_2]$, $[a_2, a_3]$, $[a_3, a_4]$, $[a_4, a_5]$, and is constant on $(-\infty, a_1]$ and on $[a_5, \infty)$. [**4**]

(c) Determine the values of all fixed points of f. [**5**]

(d) Does f have any periodic points that are not fixed points? Justify your answer. [**5**]

(e) Evaluate the set $\{n \in \mathbb{N} : f \text{ has no point of prime period } n\}$, being careful to justify your answer. [**10**]

Question 4 [25 marks]. Given an iterated function system defined by the two maps $\phi_1(x) = (x + a_3)/10$ and $\phi_2(x) = (x + a_5)/10$, define $\Phi(A) = \phi_1(A) \cup \phi_2(A)$ for all sets $A \subset \mathbb{R}$, and let C_k denote $\Phi^k([0, 1])$ for $k \geq 0$.

(a) Write down a_3 and a_5, and determine the sets C_1 and C_2. [**4**]

(b) If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k. [**3**]

(c) What is the common length of each of the N_k closed intervals whose disjoint union equals C_k? [**3**]

(d) Compute the box dimension of $C = \cap_{k=0}^{\infty} C_k$, being careful to justify your answer. [**5**]

(e) Give a description of the members of C in terms of the digits of their decimal expansion. [**5**]

(f) If $f : C \to C$ is defined by $f(x) = 10x \pmod 1$ then find a point $x \in C$ which has prime period 3 under f. [**5**]

End of Paper.