Main Examination period 2019

MTH6107: Chaos and Fractals

Duration: 2 hours

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You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

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Examiners: A. Clark, O. Jenkinson

Question 1. [25 marks]

(a) Consider a function $f : \mathbb{R} \to \mathbb{R}$.

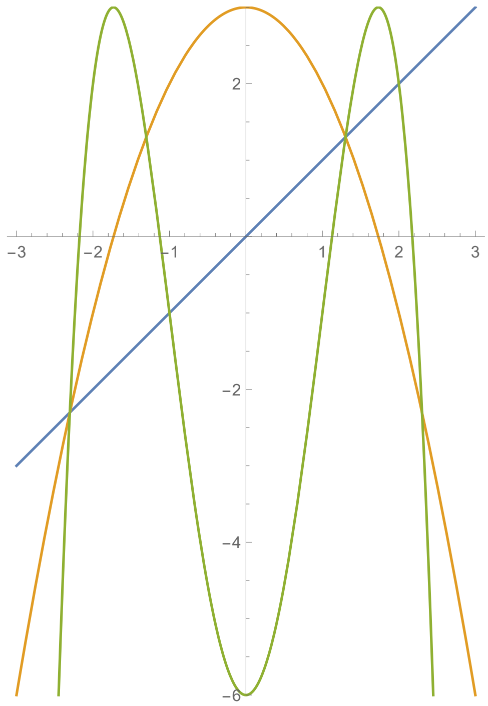
(i) What does it mean to say that a point $x_0 \in \mathbb{R}$ is a **fixed point** for f? **[1]**

(ii) What does it mean to say that a point $x_0 \in \mathbb{R}$ is a **periodic point** for f? **[1]**

(iii) How is the **prime period** of a periodic point defined? **[1]**

(iv) What does it mean to say that a point $x_0 \in \mathbb{R}$ is an **eventually periodic point** for f? **[1]**

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = -x^2 + 3$. Below is graphed $y = f(x)$, $y = f^2(x)$ and $y = x$.



(i) Compute the values of all fixed points of f. **[4]**

(ii) Using the graph as a guide, find all points of prime period 2. **[4]**

(iii) Find an eventually periodic point that is not periodic, or give a reason why there is none. **[3]**

(iv) For each 2–cycle of f, determine whether the 2–cycle is attracting, repelling or neither attracting nor repelling. Justify your answer. **[4]**

(c) If the function $g : \mathbb{R} \to \mathbb{R}$ is continuous and if for some point $x \in \mathbb{R}$ we have that

$$\lim_{n\to\infty} g^n(x) = y,$$

show that y is a fixed point of g. **[6]**

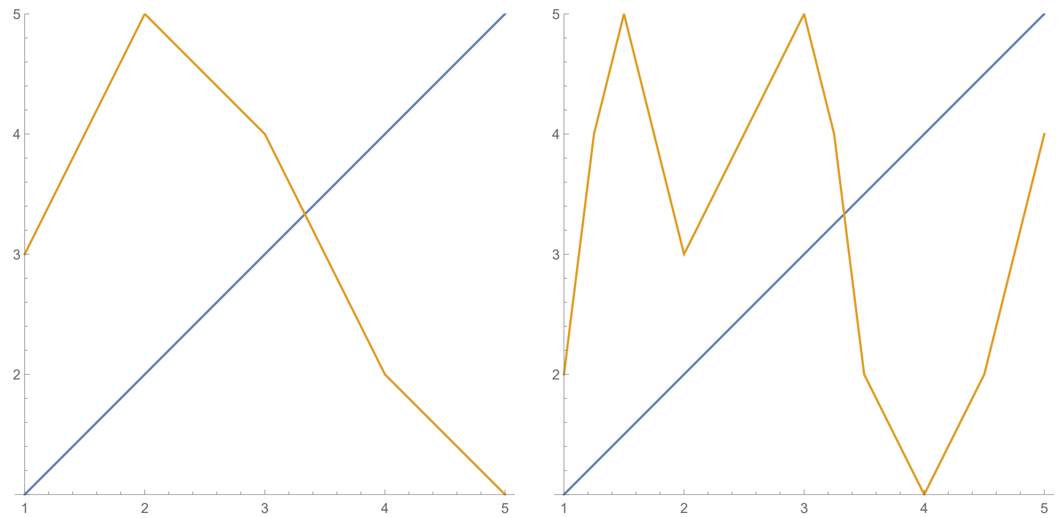
Question 2. [25 marks]

(a) Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 2x - x^2$.

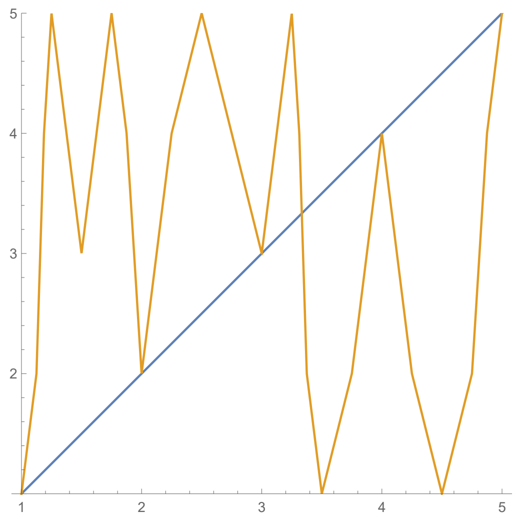
- (i) Sketch the graph of f. **[2]**
- (ii) Compute the values of all fixed points, and determine whether these points are attracting or repelling (or neither). **[6]**
- (iii) Using the graph or otherwise, find the basin(s) of attraction of the attracting fixed point(s). **[6]**

(b) State **Sharkovskii's Theorem** concerning the existence of periodic points of specified prime periods for a continuous map f from $\mathbb{R}$ to itself. **[5]**

(c) For a continuous function $f : [1,5] \to [1,5]$ the graphs of $y = f^n(x)$ for $n = 1, 3$ and 5 together with $y = x$ are pictured below. Using Sharkovskii's Theorem determine for which n, f has points of prime period n. **[6]**



Graph of $y = f(x)$ (left) and graph of $y = f^3(x)$ (right).



Graph of $y = f^5(x)$.

Question 3. [25 marks]

(a) Let $f_\mu : \mathbb{R} \to \mathbb{R}$ denote the logistic map, defined by $f_\mu(x) = \mu x(1-x)$, for real positive values of the parameter μ.

(i) Explain the mechanism by which an attracting cycle of period 2 is 'born' as μ passes through the value $\mu = 3$ **[4]**

(ii) Describe the **period-doubling bifurcation cascade** which follows as μ is further increased. **[6]**

(b) For $a \in \mathbb{R}$ define the family of maps $g_a : \mathbb{R} \to \mathbb{R}$ by $g_a(x) = -x^2 + a$.

(i) Solving the equation $g_a^2(x) = x$ algebraically yields (in addition to two solutions corresponding to fixed points) the two solutions:

$$x = \frac{1+\sqrt{4a-3}}{2}, \frac{1-\sqrt{4a-3}}{2}.$$

For which values of a does g_a have a cycle of prime period 2? **[4]**

(ii) For $\frac{3}{4} < a < \frac{5}{4}$ show that g_a has an attracting 2–cycle. **[5]**

(iii) Describe the basin of attraction for the attracting 2–cycle of g_1. **[6]**

Question 4. [25 marks] Let

$$S = \{(s_1, s_2, ...) \ : \ s_i \in \{0,1\} \text{ for all } i\}$$

and let $\sigma : S \to S$ be the shift map $\sigma((s_1, s_2, s_3, ...)) = (s_2, s_3, ...)$. Now let C be the middle thirds Cantor set

$$C = \left\{ \sum_{i=1}^{\infty} \frac{d_i}{3^i} \in [0,1] \ : \ d_i \in \{0,2\} \text{ for all } i \right\}$$

and let $t : C \to C$ be the continuous function defined by

$$t\left(\sum_{i=1}^{\infty} \frac{d_i}{3^i}\right) = \sum_{i=1}^{\infty} \frac{d_{i+1}}{3^i}.$$

(a) Let $h : S \to C$ be defined by

$$h((s_1, s_2, ...)) = \sum_{i=1}^{\infty} \frac{2s_i}{3^i}$$

Show that $h \circ \sigma = t \circ h$. **[4]**

(b) Find all the points of prime period 2 for t and give their decimal expressions. **[5]**

(c) Explain how to construct a point $x \in C$ which has the property that the orbit of x under t is dense in C. (You are not expected to prove that x has this property.) **[5]**

(d) Define the **Lyapunov exponent** of a differentiable map $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}$, and for the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ compute the Lyapunov exponent of the point $x = 1$. **[5]**

(e) State Devaney's definition of chaos and provide an example of function that satisfies the defintion. **[6]**

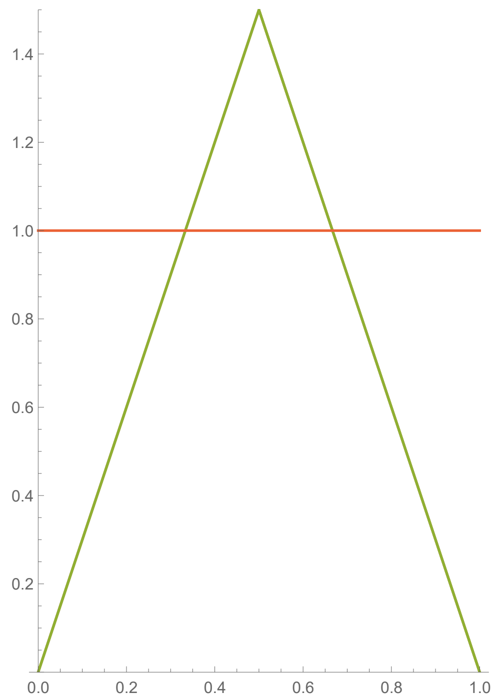
Question 5. [25 marks]

(a) Define the **box-counting dimension** of a bounded subset of $\mathbb{R}^n$, assuming it exists. **[4]**

(b) Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 3x & \text{for } x \leq 1/2 \\ 3 - 3x & \text{for } x \geq 1/2, \end{cases}$$

whose graph over the interval $[0, 1]$ is pictured below with the line $y = 1$.



Now for $i = 1, 2, \ldots$ let

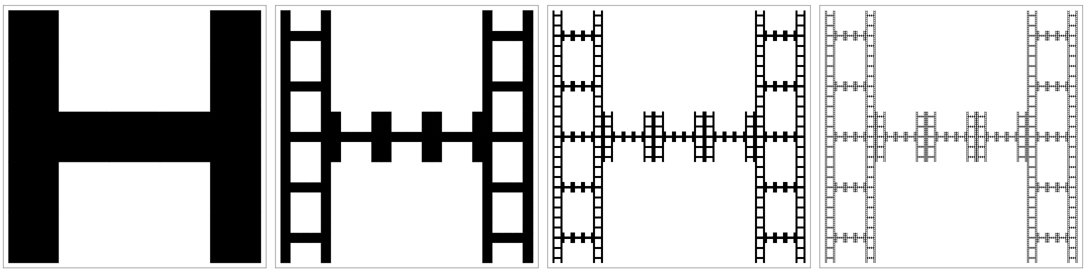
$$\Lambda(i) = \{\, x \in [0, 1] \,:\, f^n(x) \in [0, 1] \text{ for } n = 0, 1, ..., i \,\} \text{ and let}$$

$$\Lambda(f) = \{\, x \in [0, 1] \,:\, f^n(x) \in [0, 1] \text{ for all } n \geq 0 \,\}.$$

(i) Describe the two sets $\Lambda(1)$ and $\Lambda(2)$. **[4]**

(ii) Find the box counting dimension of $\Lambda(f)$. **[3]**

(c) For this problem we will consider a set H in $\mathbb{R}^2$ constructed via a recursive process similar to the Sierpinski Carpet. We begin in the initial Step 0 with the unit square $[0,1] \times [0,1]$. In Step 1 we subdivide the original square into 25 equal squares and remove the middle three squares in the first, second, fourth and fifth row of squares, and we do not remove any squares from the third row. In Step 2, we subdivide each of the remaining squares into 25 equal squares and remove the middle three squares in the first, second, fourth and fifth row of squares, and we do not remove any squares from the third row. In general, going from Step n to Step $n+1$ we subdivide each of the remaining squares into 25 equal squares and remove the middle three squares in the first, second, fourth and fifth row of squares, and we do not remove any squares from the third row. H is the set that is formed by the intersection of all sets from each stage. Below we picture Step 1 through Step 4 in the construction of H.



(i) Find the (Lebesgue) measure of H. **[5]**

(ii) Find the box counting dimension of H. **[7]**

(iii) Is H a fractal? Justify your answer. **[2]**

End of Paper.