Queen Mary University of London

Main Examination period 2018

MTH6107 / MTH6107P: Chaos and Fractals

Duration: 2 hours

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Examiners: V. Anagnostopoulou, O. Jenkinson

Question 1. [25 marks]

(a) Consider a function $f : \mathbb{R} \to \mathbb{R}$.

(i) What does it mean to say that a point $x_0 \in \mathbb{R}$ is a **fixed point** for f? **[1]**

(ii) What does it mean to say that a point $x_0 \in \mathbb{R}$ is a **periodic point** for f? **[1]**

(iii) How is the **prime period** of a periodic point defined? **[1]**

(iv) What does it mean to say that a point $x_0 \in \mathbb{R}$ is a **pre-periodic point** for f? **[2]**

(v) Prove that if f is invertible then every pre-periodic point is a periodic point. **[5]**

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 - 3x + 2$.

(i) Compute the values of all fixed points and points of prime period 2 of f. **[7]**

(ii) Find whether the fixed point(s) and period two orbit(s) are attracting or repelling (or neither). **[6]**

(iii) Find a pre-periodic point that is not a periodic point, or give a reason why such a point does not exist. **[2]**

Question 2. [25 marks]

(a) Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 2x^3 - x^2$.

(i) Compute the values of all fixed points, and determine whether these points are attracting or repelling (or neither). **[9]**

(ii) Sketch the graph of f. **[2]**

(iii) For this function f, which points lie in the basin(s) of attraction of the attracting fixed point(s)? [Give reasons but a formal proof is not expected]. **[5]**

(b) (i) State Sharkovskii's Theorem concerning the existence of periodic orbits of specified prime periods for a continuous map f from $\mathbb{R}$ to itself. **[6]**

(ii) For which of the following numbers n is it true that the existence of a periodic orbit of prime period n implies the existence of periodic orbits of prime period equal to all of the other three values?

$$n = 16, 18, 40, 56.$$ **[3]**

Question 3. [25 marks]

(a) Let $f_\mu : [0,1] \to [0,1]$ denote the logistic map, defined by $f_\mu(x) = \mu x(1-x)$, for values of the parameter $\mu \in [0,4]$.

(i) Show that for $\mu > 1$ the map f_μ has a fixed point in the open interval $(0,1)$. Show that this point is an attractor when $1 < \mu < 3$ and a repeller when $\mu > 3$. **[5]**

(ii) For the logistic map, briefly describe what is meant by the **period-doubling bifurcation cascade**. **[3]**

(b) Let $D : [0,1) \to [0,1)$ denote the doubling map $D(x) = 2x \pmod 1$, and let σ denote the shift map on the space of all one-sided infinite binary sequences.

(i) List all the periodic orbits of D which have prime periods 2 and 3. **[3]**

(ii) Write down the binary digit expansions for each periodic point in (i). **[4]**

(iii) How many orbits of prime period 6 does the shift map σ have? **[3]**

(iv) Consider the binary digit sequence $\overline{0001001}$. Which real number in $[0,1)$ has this as its binary representation? What is the prime period of this point under the doubling map D? **[2]**

(c) Let f be a diffeomorphism from the real line $\mathbb{R}$ to itself. Prove that if f is order-reversing then it has exactly one fixed point. **[5]**

Question 4. [25 marks]

(a) (i) Consider intervals X and Y in $\mathbb{R}$, and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. What is meant by a **topological conjugacy** between the map f and the map g? **[3]**

(ii) Show that if f is conjugate to g, then f^k is conjugate to g^k, for all positive integers k. **[5]**

(iii) If $f : [0,1] \to [0,1]$ is defined by $f(x) = 4x(1-x)$, and $g : [-1,1] \to [-1,1]$ is defined by $g(x) = 1 - 2x^2$, use the map $h(x) = 2x - 1$ to show that f and g are topologically conjugate. **[4]**

(iv) If $f : [0,1] \to [0,1]$ is defined by $f(x) = 2x(1-x)$, and $g : [0,1] \to [0,1]$ is defined by $g(x) = x^2(1-x)$, show that f and g are not topologically conjugate. **[4]**

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Define what is meant for f to be **chaotic** (in the sense of Devaney). **[3]**

(c) State which of the following maps are chaotic. If a map is not chaotic, briefly justify your answer. **[6]**

(i) The tent map $T : [0,1] \to [0,1]$ defined by

$$T(x) = \begin{cases} 2x & \text{for } x \in [0,1/2) \\ 2-2x & \text{for } x \in [1/2,1] \,. \end{cases}$$

(ii) The logistic map $f_1 : [0,1] \to [0,1]$, defined by $f_1(x) = x(1-x)$.

(iii) The logistic map $f_4 : [0,1] \to [0,1]$, defined by $f_4(x) = 4x(1-x)$.

(iv) The map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$.

Question 5. [25 marks]

(a) Define the **box-counting dimension** of a bounded subset of $\mathbb{R}^n$, assuming it exists. **[5]**

(b) Briefly explain how the **Sierpinski square** is constructed. **[5]**

(c) Compute the box-counting dimension of the Sierpinski square, assuming it exists. **[6]**

(d) Let C denote the 'middle-$1/7$' Cantor set, the set obtained from the interval $[0, 1] \subset \mathbb{R}$ in the same way as the middle-$1/3$ Cantor set, except that at each stage of the construction the middle-$1/7$ of each remaining interval is removed. Compute the box-counting dimension of C. **[6]**

(e) Write down an iterated function system for the middle-$1/7$ Cantor set $C \subset \mathbb{R}$ (described above in part (d) of this question). **[3]**

End of Paper.