

MTH6104/MTH6104P: Algebraic structures II

Duration: 2 hours

Date and time: 10th June 2016, 2:30pm

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You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

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Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): Matthew Fayers

In this paper, we use the following notation.

- C_n denotes the cyclic group of order n .
- \mathcal{U}_n is the set of integers between 0 and n which are prime to n , with the group operation being multiplication modulo n .
- \mathcal{D}_{2n} is the group with $2n$ elements

$$1, r, r^2, \dots, r^{n-1}, s, rs, r^2s, \dots, r^{n-1}s.$$

The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

- \mathcal{Q}_8 is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j.$$

In any question, you may freely use the Coset Lemma: if f, g are elements of a group G and $H \leq G$, then $Hf = Hg$ if and only if $fg^{-1} \in H$.

Question 1.

- (a) Give the definition of a **group**. [3]

Suppose G is a group and $f, g \in G$.

- (b) Give the definition of the **powers** g^n for $n \in \mathbb{Z}$. [3]

- (c) Define what it means to say that f and g are **conjugate** in G . Give the definition of the **conjugacy class** containing g . [4]

- (d) Is it true that if f is conjugate to g , then f^3 is conjugate to g^3 ? Justify your answer. [5]

- (e) Is it true that if f^3 is conjugate to g^3 , then f is conjugate to g ? Justify your answer. [5]

- (f) Find the conjugacy class containing s in \mathcal{D}_{12} . Show your working. [5]

Question 2. Write an essay on group actions. [You should include precise definitions and statements of results, illustrated by examples and applications, and give some proofs.]

[25]

Question 3. Suppose G is a group, H is a subgroup of G and $g \in G$.

- (a) Give the definition of the **right coset** Hg . [2]
- (b) Give the definition of the **index** of H in G . [2]
- (c) Suppose X is a right coset of H in G , and $a, b, c \in X$. Prove that $ab^{-1}c \in X$. [4]
- (d) Give the definition of a **normal subgroup** of G . [2]
- (e) Give an example of a group G and a subgroup H which is not normal. [You do not need to prove anything.] [2]
- (f) Suppose N is a normal subgroup of G . Give the definition of the **quotient group** G/N . [You do not have to prove that G/N is a group, but you should prove that the group operation is well-defined.] [6]
- (g) Find a subgroup N of \mathcal{U}_{40} such that $|N| = 4$. Write down all the cosets of N in \mathcal{U}_{40} , and find the Cayley table of \mathcal{U}_{40}/N . [You do not need to prove anything.] [7]

Question 4. Suppose G and H are groups and $g \in G$.

- (a) Give the definitions of the following terms:
- the **order** of g ;
 - a **homomorphism** from G to H ;
 - an **isomorphism** from G to H ;
 - an **automorphism** of G ;
 - an **inner automorphism** of G ;
 - the **automorphism group** of G . [8]
- (b) Give an example of an automorphism of C_8 which is not the identity map. [2]
- (c) Suppose $\phi : G \rightarrow H$ is an isomorphism, and $g \in G$. Prove that $\text{ord}(\phi(g)) = \text{ord}(g)$. [You may assume that $\phi(1) = 1$ and that ϕ^{-1} is a homomorphism.] [4]
- (d) Write down a theorem describing the inner automorphism group of G in terms of the centre of G . [You do not need to prove anything.] [3]

Suppose ψ is an automorphism of \mathcal{Q}_8 satisfying $\psi(-j) = i$ and $\psi(k) = j$.

- (e) Write down $\psi(g)$ for every $g \in \mathcal{Q}_8$, and hence find the order of ψ . [5]
- (f) Is ψ an inner automorphism of \mathcal{Q}_8 ? Justify your answer. [3]

Question 5.

- (a) Give the definition of a **simple** group. [You do not need to define what a group or a normal subgroup is.] [2]
- (b) Suppose G is an abelian group of order 49. Prove that G is not simple. [You may use basic results about the orders of elements of G .] [6]
- (c) Now suppose G is a finite group and p is a prime. Give the definition of a **Sylow p -subgroup** of G . [2]
- (d) Find a Sylow 2-subgroup, a Sylow 3-subgroup and a Sylow 5-subgroup of \mathcal{D}_{12} . [You do not need to prove anything.] [5]
- (e) Give precise statements of all the Sylow Theorems. [6]
- (f) Using the Sylow theorems, prove that there is no simple group of order 40. [4]

Question 6.

- (a) Give the definition of the **symmetric group** \mathcal{S}_n and of the **alternating group** \mathcal{A}_n . [5]
- (b) Suppose $g \in \mathcal{S}_n$. Explain how to write g in **disjoint cycle notation**. [3]
- (c) Explain how to use the disjoint cycle notation for g to find whether $g \in \mathcal{A}_n$. [You do not need to prove anything.] [2]
- (d) Prove that any element of \mathcal{A}_n can be written as a product of 3-cycles. [5]
- (e) Now let G be any group, and suppose $H \leq G$ and $N \triangleleft G$. Define the set NH , and prove that it is a subgroup of G . [5]
- (f) Now suppose $G = \mathcal{S}_4$ and $N = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Write down a subgroup $H \leq \mathcal{S}_4$ different from N such that $|H| = 4$, and find all the elements of NH . [You do not need to prove anything.] [5]

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