Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH6104: Algebraic structures II

Duration: 2 hours

Date and time: 28th May 2015, 2:30pm

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Answer four questions. If you answer more questions than specified, only the first four answers will be marked, except for the award of a bare pass.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): Matthew Fayers

In this paper, we use the following notation.

- C_n denotes the cyclic group of order n.
- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.
- $\mathcal{D}_{2n}$ is the group with $2n$ elements

$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$

  The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.

- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$ (with the group operation being composition).
- If p is a prime, then $\mathbb{Z}/p\mathbb{Z}$ is the set $\{0, 1, \ldots, p-1\}$, with addition and multiplication modulo p. $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ is the group of 2×2 invertible matrices with entries in $\mathbb{Z}/p\mathbb{Z}$, with the group operation being matrix multiplication.

Question 1.

(a) Give the definition of a *group*. **[3]**

Suppose G is a group and $g \in G$. In the rest of this question you may use elementary rules for manipulating powers of elements.

(b) Give the definition of the *cyclic subgroup* $\langle g \rangle$, and prove that it is a subgroup of G. **[6]**

(c) In the case where $G = \mathcal{U}_{25}$, find $\langle 6 \rangle$. **[4]**

(d) Give the definition of the *order* of g. **[2]**

(e) Suppose $\mathrm{ord}(g) = n < \infty$, and m is an integer such that $g^m = 1$. Prove that n divides m. **[5]**

(f) Suppose $\mathrm{ord}(f) = 3$, $\mathrm{ord}(g) = 2$ and $fg = gf$. What is the order of fg? Justify your answer. **[5]**

Question 2. Write an essay on group homomorphisms. *[You should include precise definitions and statements of results, illustrated by examples, and give some proofs.]* **[25]**

Question 3.

(a) Give the definition of a *normal* subgroup and the definition of a *simple* group. *[You do not need to define what a group or a subgroup is.]* **[4]**

(b) Suppose G is an abelian group of order 60. Prove that G is not simple. *[You may use basic results about the orders of elements of G.]* **[6]**

(c) Now suppose G is a finite group and p is a prime. Give the definition of a *Sylow p-subgroup* of G. **[2]**

(d) Write down a Sylow 2-subgroup, a Sylow 3-subgroup and a Sylow 5-subgroup of $\mathcal{U}_{27}$. **[6]**

(e) State Sylow's Theorem 3 concerning the number of Sylow p-subgroups of a finite group. **[3]**

(f) Using this theorem, prove that there is no simple group of order 63. **[4]**

Question 4. Suppose G is a group, H is a subgroup of G and $f, g \in G$.

(a) Define the *right coset* Hg, and the *index* of H in G. **[4]**

(b) Give a precise statement of Lagrange's Theorem. **[2]**

(c) Find all the right cosets of $\{1, r^3, rs, r^4s\}$ in $\mathcal{D}_{12}$. **[5]**

(d) Suppose H is a normal subgroup of G. Prove that $Hg = gH$ for every $g \in G$. **[5]**

(e) Hence show that $\langle(1\ 2\ 3\ 4)\rangle$ is not a normal subgroup of $\mathcal{S}_4$. **[3]**

(f) Now let

$$X = \left\{ g \in \mathcal{S}_{10} \;\middle|\; g(1) = 2 \right\}.$$

Find a subgroup H of $\mathcal{S}_{10}$ and $g \in \mathcal{S}_{10}$ such that $X = Hg$. Justify your answer. *[In particular, you should prove that H is a subgroup of $\mathcal{S}_{10}$.]* **[6]**

Question 5. Suppose G is a group.

(a) Define what it means for two elements of G to be *conjugate* in G. **[2]**

(b) Show that (1 2 3)(4 5) and (1 2 5)(3 4) are conjugate in $\mathcal{S}_5$. **[3]**

(c) Give the definition of the *centre* of G. **[2]**

(d) Find (with proof) the centre of $\mathcal{D}_{10}$. **[7]**

(e) Suppose $G/Z(G)$ is a cyclic group. Prove that G is abelian. **[5]**

(f) Suppose $g \in G$. Give the definition of the *centraliser* $C_G(g)$. **[2]**

(g) Suppose $g \in G$ but $g \notin Z(G)$. Show that $Z(G) \neq C_G(g) \neq G$. **[4]**

Question 6. Suppose G is a group and X is a set.

(a) Give the definition of an *action* of G on X. **[3]**

Suppose π is an action of G on X, and define a relation $\equiv$ on X by setting $x \equiv y$ if there is $g \in G$ such that $\pi_g(x) = y$.

(b) Prove that $\equiv$ is an equivalence relation on X. **[4]**

(c) Suppose $x \in X$. Give the definitions of the *orbit* and the *stabiliser* of x under π. **[4]**

(d) Give an example of a transitive action of C_4 on C_4. **[3]**

(e) Give a precise statement of the Orbit–Stabiliser Theorem. **[3]**

Now let $G = \mathrm{GL}_2(\mathbb{Z}/11\mathbb{Z})$. Let X be the set of non-zero column vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ with $a, b \in \mathbb{Z}/11\mathbb{Z}$. Define an action of G on X by $\pi_g(x) = gx$. (You may assume that π really is an action.)

(f) Prove that π is transitive. **[4]**

(g) Hence use the Orbit–Stabiliser Theorem to find $|G|$. **[4]**

End of Paper.