Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH5123: Differential Equations

Duration: 2 hours

Date and time: 1st of May 2014, 10:00am–12:00am

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You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): Yan Fyodorov

Question 1

(a) Find a function $f(u)$ such that the differential equation

$$f(x+y) + \ln x + \left(e^{x+y} + y^2\right) \frac{dy}{dx} = 0$$

is exact. **[5]**

(b) For the chosen $f(u)$ write down the corresponding solution in implicit form. **[11]**

(c) Consider the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad f(x,y) = \sqrt{y^2+9}, \quad y(1) = 0\,.$$

Show that the Picard-Lindelöf Theorem guarantees the uniqueness and existence of the solution to the above problem in a rectangular domain $\mathcal{D} = (|x-a| \le A,\ |y-b| \le B)$ in the xy plane, and specify the parameters a and b. Find the possible range of values of the height B of the domain $\mathcal{D}$ given that the width A of the domain satisfies $A < 1/2$. **[9]**

Question 2

(a) Find the general solution of the homogeneous ODE

$$y'' + 9y = 0\,.$$

[5]

(b) Find the general solution of the non-homogeneous ODE

$$y'' + 9y = \sin(2x)\,.$$

[11]

(c) Write down the general solution to the first order homogeneous linear ODE

$$y' = \tan(x)\, y\,.$$

[5]

(d) Solve the initial value problem for the first order linear non-homogeneous ODE

$$y' = \tan(x)\, y + 1, \quad y(0) = 2.$$

[4]

Question 3 Write down the solution to the following Boundary Value Problem (BVP) for the second order non-homogeneous differential equation

$$\frac{d^2y}{dx^2} = f(x), \quad y(0) = 0,\ y'(1) = 0$$

by using the Green's function method along the following lines:

(a) Formulate the corresponding left-end initial value problem and find its solution $y_L(x)$. **[8]**

(b) Formulate the corresponding right-end initial value problem and find its solution $y_R(x)$. **[7]**

(c) Use $y_L(x), y_R(x)$ for constructing the Green's function $G(x, s)$. **[6]**

(d) Write down the solution to the BVP in terms of $G(x, s)$ and $f(x)$ and use it to find the explicit form of the solution for $f(x) = x^2$. **[4]**

Question 4 Consider a system of two nonlinear first-order ODEs:

$$\dot{x} = -x - 3y - 3x^3, \quad \dot{y} = \frac{4}{3}x - y - \frac{1}{3}x^3 \tag{1}$$

(a) Write down in the matrix form the system obtained by linearization of the above equations around the point $x = y = 0$ and find the corresponding eigenvalues and eigenvectors. **[8]**

(b) Write down general solution of the linear system. Discuss the stability of zero solution of such a linear system and determine the value $x(t \to \infty)$. **[4]**

(c) Find the solution of the linear system corresponding to the initial conditions $x(0) = 2$, $y(0) = 0$. Determine the type of equilibrium for the system and describe in words the shape of trajectory in the phase plane corresponding to the specified initial conditions. Determine the tangent vector to the trajectory at $t = 0$. **[8]**

(d) Demonstrate how to use the function $V(x, y) = \frac{4}{3}x^2 + 3y^2$ to investigate the stability of the full non-linear system (1). **[5]**

End of Paper

Formula Sheet

- **Useful integrals**:

$$\int x^a\,dx = \frac{1}{a+1}\,x^{a+1}, \quad \forall a \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| \quad \text{for } a=-1; \quad \int \ln x\,dx = x\ln|x| - x$$

$$\int \cos x\,dx = \sin x, \quad \int \sin x\,dx = -\cos x, \quad \int \tan x\,dx = -\ln|\cos x|$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\,(a\cos bx + b\sin bx)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\,(a\sin bx - b\cos bx)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}, \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{|x-a|}{|x+a|},$$

- **Useful trigonometric formulae**:

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \cos 2x = \cos^2 x - \sin^2 x, \quad \sin 2x = 2\sin x\cos x$$

$$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B, \quad \cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$$

- **Reminder on ODEs:**

$$y' = f(ax+by+c) \quad \Rightarrow \quad z = ax+by+c; \quad y' = f\left(\frac{y}{x}\right) \quad \Rightarrow \quad y = x\,z$$

$$y' = A(x)\,y + B(x) \quad \text{is solved by variation of parameters method.}$$

It starts with finding the solution of the corresponding homogeneous equation $y' = A(x)\,y$, and proceeds through replacing the constant of integration with a function to be determined.

If the equation $P(x,y) + Q(x,y)\,\frac{dy}{dx} = 0$ is exact, its solution can be found in the form $F(x,y) = Const.$ where

$$P = \frac{\partial F}{\partial x} \quad \text{and} \quad Q = \frac{\partial F}{\partial y}$$

- For the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(a) = b$$

the Picard-Lindelöf Theorem guarantees the uniqueness and existence of the solution to the above problem in a rectangular domain $\mathcal{D} = (|x-a| \leq A,\ |y-b| \leq B)$ in the xy plane, provided (**i**) $f(x,y)$ is continuous and therefore bounded in $\mathcal{D}$ and $|\frac{\partial f}{\partial y}|$ is bounded in $\mathcal{D}$; (**ii**) the parameters A and B satisfy $A < B/M$ where $M = max_{\mathcal{D}}|f(x,y)|$.

- A particular solution of second-order non-homogeneous ODE with constant coefficients found by the Variation of Parameters method is given by

$$y_p(x) = \frac{1}{(\lambda_1 - \lambda_2)a_2}\left\{e^{\lambda_1 x}\int f(x)e^{-\lambda_1 x}dx - e^{\lambda_2 x}\int f(x)e^{-\lambda_2 x}dx\right\} \tag{2}$$

- If there exists a unique solution $y(x)$ to a non-homogeneous **boundary value** problem for ODE $\mathcal{L}(y) = a_2(x)y'' + a_1(x)y' + a_0(x) = f(x)$ in an interval $x \in [x_1, x_2]$ with linear B.C.

$$\alpha y'(x_1) + \beta y(x_1) = b_1, \quad \gamma y'(x_2) + \delta y(x_2) = b_2$$

it can be found by the Green's function method:

$$y(x) = \int_{x_1}^{x_2} G(x,s)\, f(s)\, ds, \quad G(x,s) = \begin{cases} A(s)\, y_L(x), & x_1 \le x \le s \\ B(s)\, y_R(x), & s \le x \le x_2 \end{cases}$$

where

$$A(s) = \frac{y_R(s)}{a_2(s)W(s)}, \quad B(s) = \frac{y_L(s)}{a_2(s)W(s)}, \quad W(s) = y_L(s)y_R'(s) - y_R(s)y_L'(s)$$

and $y_L(x), y_R(x)$ are solutions to the left/right initial value problems:

$$\mathcal{L}(y) = 0,\ y(x_1) = \alpha,\ y'(x_1) = -\beta; \quad \text{and} \quad \mathcal{L}(y) = 0,\ y(x_2) = \gamma,\ y'(x_2) = -\delta$$

- The orbital derivative for a Lyapunov function $V(x,y)$ is defined as:

$$\mathcal{D}_f V = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$