Queen Mary
University of London

Main Examination period 2018

MTH5109: Geometry II: Knots and Surfaces

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: A. Shao, S. Majid

Note that there is a compendium of definitions and formulae in the appendix, which you are free to use without comment.

Question 1. [12 marks] Consider the following parametric curve:

$$\gamma : \left(0, 4\sqrt{\pi}\right) \to \mathbb{R}^2, \qquad \gamma(t) = (\cos(t^2), \sin(t^2)).$$

(a) Show that γ is regular. **[4]**

(b) Compute the *signed* curvature of γ at each of its points. **[5]**

(c) *Without resorting to direct computations*, explain why the arc length of γ is 16π. **[3]**

Question 2. [14 marks] Consider the following parametric curve:

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma(t) = (e^t, t, t^2).$$

(a) Show that γ is regular. **[4]**

(b) Compute the curvature of γ at each of its points. **[5]**

(c) Compute the torsion of γ at each of its points. **[5]**

Question 3. [10 marks] Let C denote the unit circle about the origin in $\mathbb{R}^2$, and consider the following two parametrisations of C:

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (\cos t, \sin t),$$
$$\lambda : \mathbb{R} \to \mathbb{R}^2, \qquad \lambda(t) = (\sin t, \cos t).$$

(a) Do γ and λ have the same *unsigned* curvature at corresponding points (that is, at common points in $\mathbb{R}^2$)? *Briefly justify your answer without resorting to computations.* **[3]**

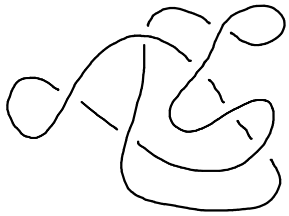
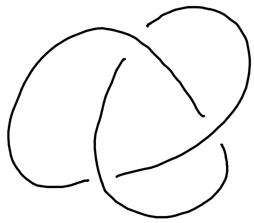
(b) Do γ and λ have the same tangent line at corresponding points (that is, at common points in $\mathbb{R}^2$)? *Briefly justify your answer without resorting to computations.* **[3]**

(c) Find the tangent line to C at the point

$$\mathbf{p} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$ **[4]**

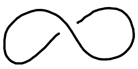
Question 4. [17 marks]

(a) Give, through drawings, a sequence of Reidemeister moves that transforms the knot diagram (i) below into the knot diagram (ii). *Indicate clearly which Reidemeister move is being used, and where it is being applied.*

(i)  (ii) 

[5]

(b) Show, *using only the definition of the Kauffman bracket* (and not the rules for how the bracket is affected by Reidemeister moves), that

B ( , x) $= -x^3.$ [4]

(c) Suppose you have a knot K. Also, suppose Alice tells you that K is achiral, while Bob tells you that its Jones polynomial satisfies

$$J(K, t) = t^2 + t^{-1}.$$

Explain how you can conclude that at least one of them is lying. [4]

(d) Give an example of a link diagram D such that its Kauffman bracket satisfies

$$B(D, x) = -(x^2 + x^{-2})^5.$$

Briefly justify your answer. [4]

Question 5. [17 marks] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let S be defined as the image of the parametric surface

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (u, v, f(u, v)).$$

Give all answers below in terms of the function f.

(a) Find the tangent plane to S at each point $\sigma(u, v) \in S$. [4]

(b) Compute the first fundamental form F^I_σ with respect to σ. [4]

(c) Find the unit normals to S at every point $\sigma(u, v) \in S$. [4]

(d) Compute the second fundamental form F^{II}_σ with respect to σ. [5]

Question 6. [21 marks] Let S be defined as the image of the parametric surface

$$\sigma : \mathbb{R} \times (0, 1) \to \mathbb{R}^3, \qquad \sigma(u, v) = (v \cos u, v \sin u, 1 - v).$$

(a) Sketch S. In addition, indicate some curves of constant u and v on your sketch. **[4]**

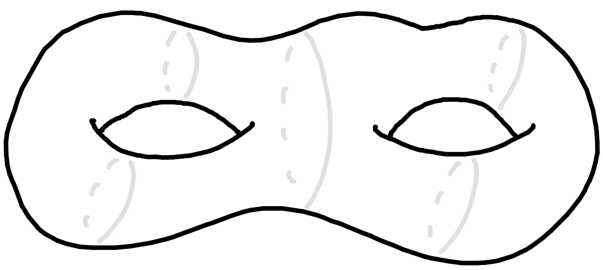
(b) Argue *from the form of the curves in your answer to (a)* (*and without doing any further computations*) that the Gauss curvature of S vanishes everywhere. **[4]**

(c) Compute the second fundamental form F^{II}_σ with respect to σ. **[6]**

(d) Compute the Weingarten matrix W_σ with respect to σ. **[4]**

(e) Compute the principal curvatures of S at any $\mathbf{p} = \sigma(u, v) \in S$ with respect to σ. In particular, confirm that the Gauss curvature of S vanishes everywhere. **[3]**

Question 7. [9 marks] Consider the surface S given by the following drawing:



(a) Find the surface integral

$$\int_S \mathcal{K}\, dA,$$

where $\mathcal{K}$ is the Gauss curvature of S. **[3]**

(b) Use part (a) to conclude that there is some point of S at which the Gauss curvature must be strictly negative. **[3]**

(c) Show that there is some point of S at which one of the principal curvatures must be strictly negative. **[3]**

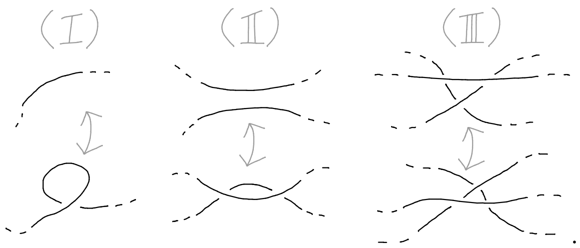
End of Paper – An appendix of 3 pages follows.

Partial list of definitions and formulas

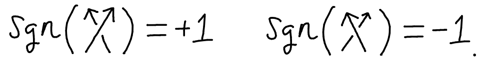
- **Parametric curve**: Smooth function $\gamma : I \to \mathbb{R}^n$, with I an open interval.
- A parametric curve $\gamma : I \to \mathbb{R}^n$ is **regular** iff $|\gamma'(t)| \neq 0$ for every $t \in I$.
- **Curve**: Roughly, a parametric curve, except reparametrisations are considered as the same.
- **Oriented curve**: Roughly, a curve with a choice of orientation.
- **Tangent line** of a parametric curve $\gamma : I \to \mathbb{R}^n$:

  Using tangent vectors: $T_\gamma(t) = \{s\gamma'(t)|_{\gamma(t)} \mid s \in \mathbb{R}\}$,

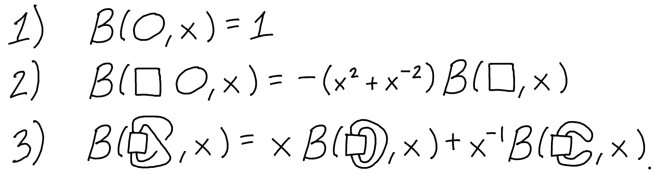
  As a set of points: $\mathcal{T}_\gamma(t) = \{\gamma(t) + s\gamma'(t) \mid s \in \mathbb{R}\}$.
- **Path integral** of a curve C, represented by a parametric curve $\gamma : (a, b) \to \mathbb{R}^n$:
$$\int_C F\,ds = \int_a^b F(\gamma(t))|\gamma'(t)|dt.$$
- **Arc length** of a curve C:
$$L(C) = \int_C 1\,ds.$$
- **Curvature** of a regular parametric curve γ, at $\gamma(t)$:
$$\kappa|_{\gamma(t)} = \frac{1}{|\gamma'(t)|}\left|\frac{d}{dt}\left[\frac{\gamma'(t)}{|\gamma'(t)|}\right]\right|.$$
- Formulas for curvature and signed curvature, respectively, for a regular plane curve γ:
$$\kappa|_\gamma = \frac{|x'y'' - y'x''|}{|\gamma'|^3}, \qquad \kappa_s|_\gamma = \frac{x'y'' - y'x''}{|\gamma'|^3}, \qquad \gamma(t) = (x(t), y(t)).$$
- Angle change formula for a plane curve C, and the **winding number** of a closed plane curve C:
$$\Delta\theta = \int_C \kappa_s\,ds, \qquad N(C) = \frac{1}{2\pi}\int_C \kappa_s\,ds.$$
- Formula for curvature and **torsion**, respectively, of a regular space curve γ:
$$\kappa|_\gamma = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3}, \qquad \tau|_\gamma = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{|\gamma' \times \gamma''|^2} \text{ (when } \kappa|_\gamma \neq 0\text{)}.$$
- **Knots**: Roughly, simple closed space curves (or knot diagrams), except that knot-equivalent curves (or diagrams) are considered to be the same knot.
- **Reidemeister moves**:



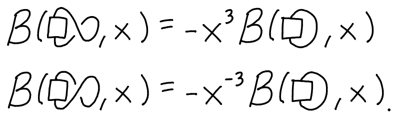
- **Reidemeister theorem**: Two knot diagrams are knot-equivalent if and only if one can be transformed into the other via a sequence of **Reidemeister moves**.
- A knot is **tricolourable** iff its segments can be 3-coloured so that (i) all three colours are used somewhere, and (ii) at each crossing, either one or all three colours are used.
- A knot is **chiral** iff its mirror image is the same knot, and **achiral** iff it is not chiral.
- **Writhe** of a knot diagram: sum of the signatures of all its crossings, where



- **Kauffman bracket** of a link diagram:



- The Kauffman bracket is unchanged under Type II and III Reidemeister moves, while



- **Jones polynomial** of a knot:
$$J(K,t) = \left(-t^{\frac{1}{4}}\right)^{3\cdot W(K)} B\left(K, t^{\frac{1}{4}}\right), \qquad W = \text{writhe.}$$

- If K is a knot, and $\tilde{K}$ its mirror image, then
$$J(\tilde{K}, t) = J(K, t^{-1}).$$

- **Parametric surface**: Smooth function $\sigma : U \to \mathbb{R}^n$, with $U \subseteq \mathbb{R}^2$ being **open** (i.e. has no boundary points) and **connected** (i.e. any $p, q \in U$ joined by a curve in U).

- A parametric surface $\sigma : U \to \mathbb{R}^n$ is **regular** iff $\partial_u \sigma(u,v)$ and $\partial_v \sigma(u,v)$ are linearly independent for all $(u,v) \in U$. When $n = 3$, then σ is regular if and only if $|\partial_u \sigma \times \partial_v \sigma| \neq 0$ everywhere.

- **Surface**: Roughly, a 2-dimensional object created by gluing together parametric surfaces (and without allowing self-intersections).

- For a surface $S \subseteq \mathbb{R}^n$, a parametrisation σ of S, and a point $\mathbf{p} \in \sigma(u_0, v_0) \in S$, we define the **tangent plane** at $\mathbf{p}$ of S as follows:

  Using tangent vectors: $T_{\mathbf{p}}S = \{a \cdot \partial_u \sigma(u_0,v_0)|_{\mathbf{p}} + b \cdot \partial_v \sigma(u_0,v_0)|_{\mathbf{p}} \mid a, b \in \mathbb{R}\}$,

  As a set of points: $\mathcal{T}_{\mathbf{p}}S = \{\mathbf{p} + a \cdot \partial_u \sigma(u_0,v_0) + b \cdot \partial_v \sigma(u_0,v_0) \mid a, b \in \mathbb{R}\}$.

- Given a surface $S \subseteq \mathbb{R}^3$ and $\mathbf{p} \in S$, we say that $N|_{\mathbf{p}}$ (where $N \in \mathbb{R}^3$) is a **unit normal** to S at $\mathbf{p}$ iff $N|_{\mathbf{p}}$ is perpendicular to $T_{\mathbf{p}}S$, and $|N| = 1$.

- Formula for unit normals to $S \subseteq \mathbb{R}^3$ at $\mathbf{p} \in \sigma(u_0, v_0) \in S$ (where σ is a parametrisation of S):
$$\pm \left[\frac{\partial_u \sigma(u_0,v_0) \times \partial_v \sigma(u_0,v_0)}{|\partial_u \sigma(u_0,v_0) \times \partial_v \sigma(u_0,v_0)|} \right] \Bigg|_{\mathbf{p}}.$$

- A surface $S \subseteq \mathbb{R}^3$ is **orientable** iff one can choose a unit normal $N|_{\mathbf{p}}$ at each $\mathbf{p} \in S$ in a way such that $N|_{\mathbf{p}}$ varies smoothly with $\mathbf{p}$.

- The **first fundamental form** of a surface $S \subseteq \mathbb{R}^n$ with respect to a parametrisation σ:
$$F^I_\sigma(u,v) = \begin{bmatrix} \partial_u \sigma(u,v) \cdot \partial_u \sigma(u,v) & \partial_u \sigma(u,v) \cdot \partial_v \sigma(u,v) \\ \partial_v \sigma(u,v) \cdot \partial_u \sigma(u,v) & \partial_v \sigma(u,v) \cdot \partial_v \sigma(u,v) \end{bmatrix}.$$

- For a surface $S \subseteq \mathbb{R}^n$ and injective parametrisation $\sigma : U \to S$, the **surface area** of $\sigma(U)$ of σ is
$$\mathcal{A}(\sigma(U)) = \iint_U \sqrt{F^I_\sigma(u,v)}\, du dv.$$
  Moreover, when $n = 3$,
$$\sqrt{F^I_\sigma(u,v)} = |\partial_u \sigma(u,v) \times \partial_v \sigma(u,v)|.$$

- Given a surface $S \subseteq \mathbb{R}^n$, an injective parametrisation $\sigma : U \to S$, and a smooth function $G : S \to \mathbb{R}$, we define the **surface integral** of G over $\sigma(U)$ by
$$\mathcal{A}(\sigma(U)) = \iint_U G(\sigma(u,v)) \sqrt{F^I_\sigma(u,v)}\, du dv.$$

- **Second fundamental form** of a surface $S \subseteq \mathbb{R}^3$ with respect to a parametrisation σ:
$$F^{II}_\sigma(u,v) = \begin{bmatrix} \partial_{uu}\sigma(u,v)\cdot N_\sigma(u,v) & \partial_{uv}\sigma(u,v)\cdot N_\sigma(u,v) \\ \partial_{vu}\sigma(u,v)\cdot N_\sigma(u,v) & \partial_{vv}\sigma(u,v)\cdot N_\sigma(u,v) \end{bmatrix},$$
$$N_\sigma(u,v) = \frac{\partial_u\sigma(u,v)\times\partial_v\sigma(u,v)}{|\partial_u\sigma(u,v)\times\partial_v\sigma(u,v)|}.$$
- **Weingarten matrix** of a surface $S \subseteq \mathbb{R}^3$ with respect to a parametrisation σ:
$$\mathcal{W}_\sigma(u,v) = F^I_\sigma(u,v)^{-1}F^{II}_\sigma(u,v).$$
- Given a surface $S \subseteq \mathbb{R}^3$, a parametrisation σ of S, and a point $\mathbf{p} = \sigma(u,v) \in S$:
  - **Principal curvatures** of S at $\mathbf{p}$ (with respect to σ): eigenvalues $\kappa_1|_\mathbf{p}$, $\kappa_2|_\mathbf{p}$ of $\mathcal{W}_\sigma(u,v)$.
  - **Mean curvature** of S at $\mathbf{p}$ (with respect to σ):
$$H|_\mathbf{p} = \frac{1}{2}(\kappa_1|_\mathbf{p} + \kappa_2|_\mathbf{p}) = \frac{1}{2}\operatorname{tr}\mathcal{W}_\sigma(u,v).$$
  - **Gauss curvature** of S at $\mathbf{p}$:
$$\mathcal{K}|_\mathbf{p} = \kappa_1|_\mathbf{p}\cdot\kappa_2|_\mathbf{p} = \det\mathcal{W}_\sigma(u,v).$$
- Additional formulas for principal curvatures:
$$\kappa_1|_\mathbf{p}, \kappa_2|_\mathbf{p} = H|_\mathbf{p} \pm \sqrt{(H|_\mathbf{p})^2 - \mathcal{K}|_\mathbf{p}}.$$
- **Gauss–Bonnet Theorem**: Let $S \subseteq \mathbb{R}^3$ be a compact surface. Then,
$$\int_S \mathcal{K}\,dA = 4\pi(1-g_S),$$
where $\mathcal{K}$ is the Gauss curvature of S, and g_S is the genus of S.

End of Appendix.