Queen Mary
University of London

Main Examination period 2019

MTH5105: Differential and Integral Analysis

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: Huy T. Nguyen and Xin Li

Question 1. [25 marks]

(a) Let $f : (a, b) \to \mathbb{R}$ be a real valued function. State the definition for f to be **differentiable** at a point $x \in (a, b)$. **[5]**

(b) Consider the following function, $g : \mathbb{R} \to \mathbb{R}$, given by

$$g(x) = x^3.$$

Using the definition of derivative, compute the derivative of g. **[5]**

(c) Consider the following function, $h : \mathbb{R} \to \mathbb{R}$,

$$h(x) = \begin{cases} 0, & x \leq 0, \\ e^{-1/x}, & x > 0. \end{cases}$$

Show that h is differentiable on $\mathbb{R}$. **[5]**

(d) Compute the following limits (with full justification) **[5]**

(i) $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1-x}}{x}$,

(ii) $\lim_{x \to 0} \frac{\exp(x) - 1 - x}{x^2}$.

(e) Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = f(x) \quad \forall x \in \mathbb{R}$$

and $f(0) = 1$. Using the property above, show that $f(x)f(-x) = 1$ and that $f(x) \neq 0$ for all $x \in \mathbb{R}$. **[5]**

Question 2. [25 marks]

(a) State the definition of a **uniformly continuous function**. **[5]**

(b) Prove that $f(x) = x^2$ is uniformly continuous on $[0, 1]$. **[5]**

(c) State the **Mean Value Theorem**. **[5]**

(d) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Show that if $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing. Is the converse statement true? If so, prove the statement, if not, give a counterexample and show that it is a counterexample. **[10]**

Question 3. [25 marks]

(a) State **Taylor's Theorem**. **[5]**

(b) Let $h : \mathbb{R}\backslash\{1\} \to \mathbb{R}$ be the function given by

$$h(x) = \frac{1}{1-x}.$$

(i) Using any correct method, show that the Taylor series of h about $x = 0$ is given by

$$\sum_{k=0}^{\infty} x^k$$

and find its radius of convergence. **[7]**

(ii) Compute the derivative of h. **[2]**

(iii) Hence, using (ii) or otherwise, find the Taylor series and radius of convergence for

$$g(x) = \frac{1}{(1-x)^2}$$

about $x = 0$. **[6]**

(c) Let $f(x)$ be twice differentiable in the interval $[a, b]$ and suppose that $f''(x) \geq 0$ for every value of x. If x_0 is any point in the interval, the tangent line at x_0 is given by $y_0 = f(x_0) + f'(x_0)(x - x_0)$. Show that f always lies above its tangent line, that is $f(x) - y_0 \geq 0$ for any x. **[5]**

Question 4. [25 marks]

(a) State the **Fundamental Theorem of Calculus**. **[5]**

(b) Let $f : [a, b] \to \mathbb{R}$ denote a continuous function and let F, G be antiderivatives of f. Show that F and G differ by a constant. **[5]**

(c) Consider the functions $f_\alpha(x) = x^\alpha$ for $x \in (0, 1]$ and $\alpha \in \mathbb{R}$. Find the anti-derivatives of f_α and compute $\int_0^1 f_\alpha(x)dx$ for the values that integral exists (give full justification). **[5]**

(d) Consider the function $h : [0, 1] \to \mathbb{R}, h(x) = x$.

(i) Show that h is Riemann integrable. **[2]**

(ii) Show that the lower sum $L(h, P_n)$ of h for the equidistant partition

$$P_n = \left\{ x_0 = 0, \cdots, x_k = \frac{k}{n}, \cdots, x_n = 1 \right\}$$ **[5]**

satisfies $\lim_{n\to\infty} L(h, P_n) = \frac{1}{2}$.
(*You may use the formula,* $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, *or any other correct method.*)

(iii) Compute the integral $\int_0^1 h(x)dx$. **[3]**

End of Paper.