Queen Mary University of London

Main Examination period 2021 – January – Semester A

MTH5104: Convergence and Continuity

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: M. Jerrum, S. Majid

You may assume any standard properties of the functions sin(), cos(), exp() and ln(), including the fact that they are continuous.

Question 1 [20 marks]. In this question, $A, B \subseteq \mathbb{R}$ are sets of real numbers.

(a) Define what it means for $x \in \mathbb{R}$ to be the **minimum** of set A, and what it means for x to be the **infimum** of A. **[4]**

(b) Suppose A has a minimum $x \in \mathbb{R}$. Prove that x is also the infimum of A. **[4]**

(c) For each of the following sets, state whether it has a minimum and, if so, what that minimum is. Also state whether it has an infimum and, if so, what that infimum is. No explanations are required.

(i) $\{1/n : n \in \mathbb{N}\}$, (ii) $\mathbb{Z}$, (iii) $\{1/n : n \in \mathbb{Z} \setminus \{0\}\}$, (iv) $\emptyset$. **[8]**

(d) Give examples of sets A and B such that neither A nor B has an infimum, but $A \cap B$ does have an infimum. Briefly explain your answer. **[4]**

Question 2 [20 marks]. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Write down quantifier expressions that express the following conditions:

(i) the sequence $(x_n)_{n=1}^{\infty}$ **converges to** 0;

(ii) the sequence $(x_n)_{n=1}^{\infty}$ **tends to** ∞. **[4]**

In the remainder of the question, $(x_n)_{n=1}^{\infty}$ **is a sequence converging to** 0.

(b) Which of the following statements are true and which are false? No explanation is required.

(i) $((-1)^n x_n)_{n=1}^{\infty}$ converges to 0.

(ii) $(x_{n+1} - x_n)_{n=1}^{\infty}$ converges to 0.

(iii) If, in addition, $x_n \neq 0$ for all $n \in \mathbb{N}$ then $(1/x_n)_{n=1}^{\infty}$ tends to ∞.

(iv) If, in addition, $x_n \geq 0$ for all $n \in \mathbb{N}$ then $(x_n^{1/n})_{n=1}^{\infty}$ converges to 0.

(v) $(|x_n|)_{n=1}^{\infty}$ converges to 0. **[10]**

(c) Assume $x_n \geq 0$, for all $n \in \mathbb{N}$. Prove, **directly from the definition**, that $(\sqrt{x_n}\,)_{n=1}^{\infty}$ converges to 0. **[6]**

Question 3 [20 marks]. For each of the sequences $(x_n)_{n=1}^{\infty}$ defined in parts (a)–(e), decide if the **sequence** converges and, if so, to what value. Justify your answers. You may appeal to any of the results covered in the module, provided you indicate which you are using.

(a) $x_n = \dfrac{1}{n}\cos(n\pi)$. **[4]**

(b) $x_n = \left(1 + \dfrac{1}{n}\right)\cos(n\pi)$. **[4]**

(c) $x_n = \dfrac{a_2n^2 + a_1n + a_0}{b_2n^2 + b_1n + b_0}$, where $b_2 > 0$ and $b_1, b_0 \geq 0$ are real numbers. **[4]**

(d) $x_n = \exp(1/n)$. **[4]**

(e) $x_n = \ln(1/n)$. **[4]**

Question 4 [12 marks].

(a) Let $(x_k)_{k=1}^{\infty}$ be a sequence of real numbers. Define what it means for the series $\sum_{k=1}^{\infty} x_k$ to **converge** to $S \in \mathbb{R}$. **[2]**

(b) Give sequences $(x_k)_{k=1}^{\infty}$ with the following properties:

(i) $\sum_{k=1}^{\infty} x_k$ converges, but $\sum_{k=1}^{\infty} kx_k$ does not.

(ii) $\sum_{k=1}^{\infty} x_k$ converges, but $\sum_{k=1}^{\infty} (-1)^k x_k$ does not.

(iii) $\sum_{k=1}^{\infty} x_k/k!$ converges, but $\sum_{k=1}^{\infty} x_k/2^k$ does not.

No explanation is required. **[6]**

(c) Which of the following series converge? Justify your answers.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{k+2}{2k^2-1}$, (ii) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(k\pi/7)}{k(k+1)}$, (iii) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k^2}{3k^2-1}\right)^k$. $+4+4+4+4$

[4+4+4]

You may use any results from the course provided you indicate which you are using.

Question 5 [20 marks].

(a) Define (using a quantifier expression) what it means to say that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at a point $a \in \mathbb{R}$. **[3]**

(b) Prove directly from the definition that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 5x + 3$ is continuous at $a = 0$.

[You may like to try δ of the form $\delta = \min\{c\varepsilon, 1\}$ for a suitable constant $c > 0$.] **[8]**

(c) Using the Intermediate Value Theorem (IVT), show that the equation $x^3 - 5x + 3 = 0$ has a solution for x in the range $(-\infty, -1]$ and another in the range $[1, \infty)$. Show explicitly that the conditions of the IVT are satisfied. **[5]**

(d) Prove that there is a real number x satisfying $x^3 - 5x + 3 = \sin(x)$. **[4]**

End of Paper.