Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH5104: Convergence and Continuity

Duration: 2 hours

Date and time: 20 May 2015, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Exam papers must not be removed from the examination room.

Examiner(s): R. Müller

You may assume any standard properties of the sine, cosine and exponential functions including that they are continuous.

Question 1 (15 marks).

(a) Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $x \in \mathbb{R}$. Define (using quantifier expressions) what it means for $(x_n)_{n=1}^{\infty}$ to *converge to* x. [**3**]

(b) For each of the following sequences, state whether the sequence converges to a limit in $\mathbb{R}$, and, if so, find the limit. Give reasons for your answers. (*You may use any results from the course provided you state clearly which result you are using.*)

(i) $x_n = 3(\sin(n))^2 \left(\frac{n+1}{2n^2}\right)$, [**3**]

(ii) $x_n = \frac{3n^2 + 7\sin(n)}{2n^2}$, [**3**]

(iii) $x_n = (-1)^n \left(\frac{2n^2+3}{2n^2}\right)$, [**3**]

(iv) $x_n = \cos\left(\pi \cdot \exp\left(\frac{4n+1}{2n^2}\right)\right)$. [**3**]

Question 2 (15 marks).

(a) Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Define (using quantifier expressions) what it means for $(x_n)_{n=1}^{\infty}$ to be a *Cauchy sequence*. [**3**]

(b) Using only the definition, but not any results proved in the course, prove that $(x_n)_{n=1}^{\infty}$ given by

$$x_n = 2 + \frac{1}{3n^2}$$

is a Cauchy sequence. [**5**]

(c) Using only the definition, but not any results proved in the course, prove that $(x_n)_{n=1}^{\infty}$ given by

$$x_n = \sum_{k=1}^{n} \frac{3}{k}$$

is *not* a Cauchy sequence. [**7**]

Question 3 (15 marks).

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $a \in \mathbb{R}$. Define (using quantifier expressions) what it means for f to be *continuous* at the point a. **[3]**

(b) For each of the following functions, state whether they are continuous at $a = 0$ and *prove* your answers.

(i) $f(x) = x^2 + 2,$ **[4]**

(ii) $f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q}, \\ -5x & \text{if } x \notin \mathbb{Q}, \end{cases}$ **[4]**

(iii) $f(x) = \begin{cases} 2x+1 & \text{if } x \geqslant 0, \\ -5x & \text{if } x < 0. \end{cases}$ **[4]**

Question 4 (15 marks).

(a) Given a sequence $(x_k)_{k=1}^{\infty}$ of real numbers and $S \in \mathbb{R}$, what does it mean to say that the sum $\displaystyle\sum_{k=1}^{\infty} x_k$ exists and equals S? **[3]**

(b) Which of the following sums exist? Briefly justify your answers.

$$(i)\ \sum_{k=1}^{\infty} \frac{3}{\sqrt{k}}, \qquad (ii)\ \sum_{k=1}^{\infty} \frac{\sin(k)}{k^4}, \qquad (iii)\ \sum_{k=1}^{\infty} \frac{1}{5k^2 - 2k}.$$

(*You may use any results from the course provided you state clearly which result you are using.*) **[6]**

(c) Compute the value of the sum $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k(k+2)}$. **[6]**

Question 5 (15 marks).

(a) State the *Intermediate Value Theorem*. **[3]**

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^4 - x^2 + x - 2$. Prove that the equation $f(x) = 0$ has at least two different solutions in the interval $[-2, 2]$. **[6]**

(c) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying $f(0) = f(1) = 0$. Prove that there exists a number $c \in [0, \frac{1}{2}]$ satisfying $f(c) = f(c + \frac{1}{2})$. **[6]**

Question 6 (10 marks). For each of the following statements, state whether it is true or false. Justification is *not* required.

In the following, the sequence $(x_k)_{k=0}^{\infty}$ is defined by $x_k = (-1)^k \frac{1}{2^k}$.

(a) The sequence $(x_k)_{k=0}^{\infty}$ has a subsequence which is monotonically increasing. **[2]**

(b) The sequence $(y_k)_{k=0}^{\infty}$ defined by $y_k = \cos(x_k)$ converges to 0. **[2]**

(c) The sum $\displaystyle\sum_{k=0}^{\infty} x_k$ exists and has value $\frac{1}{3}$. **[2]**

(d) There is a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that the sum $\displaystyle\sum_{k=0}^{\infty} x_{\phi(k)}$ exists and has value 3. **[2]**

(e) The sum $\displaystyle\sum_{k=0}^{\infty} (x_k)^2$ does not exist. **[2]**

Question 7 (15 marks).

(a) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions, $f_n : [0,1] \to \mathbb{R}$, and let $f : [0,1] \to \mathbb{R}$ be another function. Recall that we say that $(f_n)_{n=1}^{\infty}$ converges *pointwise* to f if

$$\forall x \in [0,1]\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ :\ |f_n(x) - f(x)| < \varepsilon. \qquad (1)$$

Define (using a quantifier expression similar to (1)) what it means for $(f_n)_{n=1}^{\infty}$ to converge *uniformly* to f. **[3]**

(b) Prove that the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = \dfrac{nx}{1+n^2x^2}$ converges pointwise to $f : [0,1] \to \mathbb{R}$ given by $f(x) = 0$. **[5]**

(c) For f_n and f as in part (b), prove that $(f_n)_{n=1}^{\infty}$ does *not* converge uniformly to f. (*Hint: Look at the value of* $f_n(\frac{1}{n})$.) **[7]**

End of Paper.