Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH5103: Complex Variables

Duration: 2 hours

Date and time: 13th May 2015, 10:00 –12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Exam papers must not be removed from the examination room.

Examiner(s): S. Beheshti

Question 1. (a) Find all solutions $z \in \mathbb{C}$ of the equation $(z-i)^4 - 81 = 0$. [**5**]

(b) Is the mapping given by $z \mapsto w = iz^2 + 5$ a Möbius Transformation? Provide a definition of Möbius Transformation to justify your answer. What set of points in the z-plane is mapped by this transformation to the upper half of the w-plane ($\Im w > 0$)? Your answer should include a sketch of the z-plane with an appropriately shaded region. [**5**]

(c) Let $f = u + iv$ be a complex-valued function of a complex variable $z = x + iy$. Write down the *Cauchy-Riemann equations* satisfied by the real and imaginary parts u and v of f and state the conditions under which f is guaranteed to be complex differentiable at z_0. [**5**]

(d) Find the set of points at which the function $f(x+iy) = x^2 - y^2 - y + ix(1-2y)$ is differentiable and compute the derivative(s) at those point(s). [**5**]

Question 2. (a) Find the Taylor series expansion of the function $f(z) = \dfrac{z}{2+3z}$ about $z_0 = 0$ and determine the radius of convergence of the series. [**6**]

(b) Is it possible to give an example of a power series centred at $z_0 = 0$ which is convergent for all $z \in \mathbb{R}$ but divergent at all other $z \in \mathbb{C}$? Give an example or explain why this is not possible. [**6**]

Question 3. Consider the function $f(z) = \dfrac{1}{z(z-3)}$.

(a) Find the Laurent series

$$\sum_{n=0}^{\infty} a_n (z-3)^n + \sum_{n=1}^{\infty} b_n (z-3)^{-n}$$

of $f(z)$ on a punctured disc centred at $z_0 = 3$ and specify the region on which the series is valid. [**6**]

(b) What type of singularity does f have at the point $z_0 = 3$? [**6**]

(c) Determine the residue of f at the point $z_0 = 3$. [**6**]

Question 4. (a) Explain what is meant by an *isolated singularity* of a complex function f. Give an example of a complex function which has a removable singularity. Give a second example of a function which has an essential singularity. Justify your answers, briefly. [**6**]

(b) Prove the following: If $f(z)$ has a pole of order m at $z_0 = 0$, then $g(z) = f(z^2)$ has a pole of order $2m$ at $z_0 = 0$. [**6**]

(c) Determine the singularities of $f(z) = \dfrac{e^z - 1}{z^3}$. For each pole determined, state the order of the pole and calculate its corresponding residue. [**6**]

Question 5. (a) How many roots (counted with multiplicity) of the polynomial $11z^{1001} + z^7 + 101z^3 + 55z^2 + 33$ lie in the unit disc $\{z \in \mathbb{C} : |z| < 1\}$? Justify your answer by stating and using Rouché's Theorem. You do *not* need to provide a proof of the theorem. [**7**]

(b) State Cauchy's Theorem. [**5**]

(c) Consider the closed, anticlockwise-oriented curve $C = C_1 \cup C_2$, comprised of the union of the two paths C_1 and C_2, given by

C_1 is the curve from $-i$ to i along the right half of the circle of radius 1 centred at 0

C_2 is the straight line segment from i to $-i$.

Draw the path given and use Cauchy's Theorem to calculate

$$\int_C \frac{2+z}{4+z^2}\, dz.$$

[**6**]

Question 6. (a) State the Residue Theorem. [**5**]

(b) Using the Residue Theorem, or otherwise, evaluate

$$\int_C \frac{2z+8}{(z^2+9)(z-1)^2}\, dz,$$

where C is the positively oriented circle of radius 2 centred at the origin. [**9**]

End of Paper.