B. Sc. Examination by course unit 2015

MTH5102: Calculus III

Duration: 2 hours

Date and time: 12 May 2015, 10:00 am

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): L. Lacasa

Question 1. (17 marks)
Let $\mathcal{C}$ be a curve in $\mathbf{R}^2$ whose parametric equation reads $\mathbf{r}(t) = (t\cos t, t\sin t),\ t > 0$, and consider the points $A = (-\pi, 0), B = (0, \pi/2), C = (0, 0)$.

(a) Justify which points from $\{A, B, C\}$ belong to the curve $\mathcal{C}$. [3]

(b) Consider the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$. Calculate the line integral of $\mathbf{F}$ over $\mathcal{C}$, between A and B. [6]

(c) Make a sketch of the vector field $\mathbf{F}(x, y)$.
According to this sketch (that is to say, without computing it), what can you say about the curl of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 0\mathbf{k}$? [4]

(d) Make a sketch of curve $\mathcal{C}$ (hint: use polar coordinates). [4]

Question 2. (6 marks)
Let $\mathcal{C}$ be the curve in $\mathbf{R}^2$ whose parametric equation reads $\mathbf{r}(t) = (\cos 2t, \sin 2t)$. Calculate the arc length of $\mathcal{C}$, between $t = 0$ and $t = \pi/2$. Give details of your calculations.

Question 3. (11 marks)

(a) State the difference between a scalar field and a vector field. [2]

(b) Explain what geometric properties we are actually measuring when we compute the divergence and the curl of a vector field close to a given point. [4]

(c) Prove that if we define the vector field $\mathbf{A} = \nabla \times \mathbf{F}$, then the flux of $\mathbf{A}$ over any closed surface is null for all $\mathbf{F}$. [5]

Question 4. (13 marks)
Let $U(x, y, z) = x^2 + 4y^2 + 9z^2$, and $\mathbf{F}(x, y, z) = (y + z)\mathbf{i} + x\mathbf{j} + (x + 2z)\mathbf{k}$.

(a) Calculate the gradient of U, and describe the surface $U = 56$. [4]

(b) Calculate the divergence and curl of $\mathbf{F}$. [4]

(c) State the definition of a scalar potential and compute a scalar potential for $\mathbf{F}$ in case this vector field admits one. [5]

Question 5. (17 marks)

(a) State, without proof, the Stokes theorem. Define the terms used and any required conditions. [4]

(b) Let $\mathbf{F}$ be the vector field given by $\mathbf{F} = (x^3 - 4y)\mathbf{i} + (2x - 3z)\mathbf{j} + (y + z^2)\mathbf{k}$. Using the Stokes theorem, or otherwise, evaluate the line integral of $\mathbf{F}$ around the unit circle $x^2 + y^2 = 1, z = 0$, traversed anticlockwise starting and finishing at the point (1, 0, 0). [8]

(c) Show, using the Stokes theorem, that the line integral of an irrotational vector field around a closed curve is null. [5]

Question 6. (10 marks)
The classic gravitational field is a model used to describe the influence that a massive body (such as the Earth) produces around itself, producing a force on another massive body which we call gravitational force. The gravitational field is a vector field that in spherical coordinates reads $\mathbf{G}(r, \theta, \phi) = -g\mathbf{e}_r$, where $g \approx 9.8$ is a constant and $\mathbf{e}_r$ is a unit vector in spherical coordinates.

(a) Express $\mathbf{G}$ in terms of the Cartesian unit vectors $\mathbf{i}$,$\mathbf{j}$,$\mathbf{k}$. [3]

(b) Express the area element d$\mathbf{S}$ in spherical coordinates. [3]

(c) Using spherical coordinates, compute the flux of gravitational field over the surface of the Earth, assuming the Earth is a sphere of radius R. [4]

Question 7. (14 marks)
Let $f(x)$ be a periodic function of period 2π defined in $(-\pi, \pi)$ by

$$f(x) = \begin{cases} -1/2 & \text{if } -\pi < x < 0 \\ 1/2 & \text{if } 0 \leq x < \pi. \end{cases}$$

(a) State, without proof, Dirichlet's theorem, and use it to compute $S(\pi)$, where $S(x)$ is the Fourier series of $f(x)$. [4]

(b) Calculate the Fourier series of $f(x)$. [10]

Question 8. (12 marks)
Consider the Laplace equation $\nabla^2\phi = 0$ over a rectangle in the XY plane, subject to some boundary conditions.

(a) State the Laplace equation in two dimensional Cartesian coordinates and show that $U(x, y) = \cosh x \cdot \sin y$ fulfils it. [3]

(b) Show that if n scalar fields $\phi_1, \phi_2, \ldots, \phi_n$ fulfil the Laplace equation, then their linear combination $\phi = \sum_{i=1}^{n} \alpha_i \phi_i$ (where $\alpha_i \in \mathbf{R}\ \forall i$) is a scalar field that also fulfils the Laplace equation. [3]

(c) Explain how you would check if a scalar field of the form

$$\phi(x, y) = (A\cos(kx) + B\sin(kx))(C\cosh(ky) + D\sinh(ky)),$$

where A, B, C, D, k are constants, is indeed the solution of the Laplace equation subject to some boundary conditions. [6]

End of Paper.