Queen Mary
University of London

Main Examination period 2019

MTH5100: Algebraic Structures I

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: S. Majid, X. Li

Question 1. [18 marks]

(a) State a test for a subset $S \subseteq R$ of a ring R to be a **subring**. **[4]**

(b) Let X be a set and $P(X)$ the set of subsets of X. State the standard ring operations and zero element for $P(X)$. (You are not asked to prove anything.) **[3]**

Now let $X = \{1, 2, 3\}$ and let $S = \{\emptyset, \{1\}, \{2, 3\}, X\}$.

(c) Prove that S is a subring of $P(X)$. **[7]**

(d) Is S an ideal of $P(X)$? Justify your answer. **[4]**

Question 2. [20 marks]

(a) Let $I \subseteq R$ be an ideal of a ring R. Show that $M_2(I)$ is an ideal of $M_2(R)$. **[6]**

(b) Is it true, for every ring R with identity, that any ideal of $M_2(R)$ is of the form $M_2(I)$ for some ideal $I \subseteq R$? You are not asked to prove anything. **[2]**

(c) Define what is meant by a **ring homomorphism** $\theta : R \to S$ between rings R, S. **[4]**

(d) Given a ring homomorphism $\theta : R \to S$, define $\ker(\theta)$ and prove that it is an ideal of R. You may assume elementary facts about rings. **[8]**

Question 3. [18 marks]

(a) Let $I \subseteq R$ be an ideal of a ring R. State the definition of the **factor ring** R/I, i.e., what are its elements and what are their addition and product operations. You are not asked to prove anything. **[6]**

(b) Using the 2nd isomorphism theorem, or otherwise, find **all** subrings of $\mathbb{Z}/8\mathbb{Z}$. Explain your reasoning. **[8]**

(c) Which of the subrings in part (b) are ideals? Briefly justify your answer. **[4]**

Question 4. [18 marks]

(a) Define what is meant by an **integral domain**. You should include a definition of what it means for an element of a ring to be a zero divisor. **[4]**

(b) Define what it means for two elements of an integral domain to be **associates** and for an element to be **irreducible**. **[6]**

(c) Define what is meant by a **unique factorisation domain**. **[6]**

(d) State an example of a unique factorisation domain which is **not** a principal ideal domain. You are not asked to prove anything. **[2]**

Question 5. [16 marks]

(a) Specify a map making the integral domain $\mathbb{Z}[\sqrt{-1}] = \{m + n\sqrt{-1} \mid m, n \in \mathbb{Z}\}$ into a Euclidean domain. You are not required to prove anything. **[2]**

Now let $I = \{m + n\sqrt{-1} \mid m, n \in \mathbb{Z},\ m + n \in 2\mathbb{Z}\}$.

(b) Prove that I is an ideal of $\mathbb{Z}[\sqrt{-1}]$. **[7]**

(c) Find an element $a \in \mathbb{Z}[\sqrt{-1}]$ such that $I = \langle a \rangle$. Justify your answer, starting with the meaning of the notation $\langle a \rangle$ from lectures. **[7]**

Question 6. [10 marks] Let $\mathbb{F}$ be a field and $\mathbb{F}[x]$ the ring of polynomials with coefficients in $\mathbb{F}$.

(a) What property of a polynomial $f \in \mathbb{F}[x]$ ensures that the factor ring $\mathbb{F}[x]/\langle f \rangle$ is a **field**? **[2]**

Now let $\mathbb{F} = \mathbb{F}_2 = \{0, 1\}$ be the field of two elements and let $f = 1 + x + x^2 \in \mathbb{F}_2[x]$.

(b) Show that the property alluded to in part (a) holds for f. **[4]**

(c) Let $\alpha = \langle f \rangle + x$ as an element of $\mathbb{F}_2[x]/\langle f \rangle$. Find α^{-1} as an element of $\mathbb{F}_2[x]/\langle f \rangle$. **[4]**

End of Paper.