B. Sc. Examination by course unit 2015

MTH5100: Algebraic Structures I

Duration: 2 hours

Date and time: 28 May 2015, 10.00h–12.00h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Exam papers must not be removed from the examination room.

Examiner(s): L. H. Soicher

Question 1. (a) Define what is meant by a *subring* of a ring R. **[3]**

(b) Define what is meant by an *ideal* of a ring R. **[3]**

(c) What is meant by a *homomorphism* from a ring R to a ring S? **[3]**

(d) What is meant by an *isomorphism* from a ring R to a ring S? **[3]**

Question 2. Let S be the subring $\{[0]_4, [2]_4\}$ of $\mathbb{Z}_4$, and let T be the subring $\{[0]_6, [3]_6\}$ of $\mathbb{Z}_6$. [You are not required to prove that these are subrings.]

(a) Is T a ring with identity? Justify your answer. **[3]**

(b) Is T a field? Justify your answer. **[3]**

(c) Is $\mathbb{Z}_6$ a field? Justify your answer. **[3]**

(d) Is S an ideal of $\mathbb{Z}_4$? Justify your answer. **[3]**

(e) Give an explicit example of a homomorphism of rings from S to T. **[3]**

(f) Is there an isomorphism of rings from S to T? Justify your answer. **[3]**

Question 3. Let X be a set and let $\mathcal{P}(X)$ denote the Boolean ring whose elements are the subsets of X, with addition being symmetric difference and multiplication being intersection. [You do not have to prove that $\mathcal{P}(X)$ is a ring.]

(a) What is the zero-element of $\mathcal{P}(X)$? **[2]**

(b) Determine coset representatives for the distinct cosets of the subring $\mathcal{P}(\{2\})$ in $\mathcal{P}(\{1, 2, 3\})$. **[6]**

Question 4. Let R be a ring and let I be an ideal of R.

(a) Define what is meant by the *factor ring* R/I. [You do *not* need to show that your definitions of addition and multiplication are well-defined.] **[3]**

(b) Consider the map $\theta : R \to R/I$, defined by $r\theta = I + r$ for all $r \in R$. Prove that θ is a surjective homomorphism of rings, with $\text{Ker}(\theta) = I$. [You may assume, without proof, that R/I is a ring.] **[8]**

(c) Give, without proof, an ideal J of the ring $\mathbb{R}[x]$ of polynomials with real number coefficients, such that $\mathbb{R}[x]/J$ is isomorphic to the field $\mathbb{C}$ of complex numbers. **[3]**

Question 5. Let $S = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$.

(a) Apply a subring test to prove that S is a subring of $\mathbb{C}$. [**5**]

(b) Explain why S is an integral domain. [**3**]

(c) Define what is meant by a *unit* in a ring with identity. [**3**]

(d) Determine, with justification, the units in S. [**5**]

(e) Is S a unique factorisation domain? Justify your answer. [**6**]

Question 6. In this question we consider the ring $\mathbb{Q}[x]$ of polynomials with rational number coefficients. [You may assume, without proof, that $\mathbb{Q}[x]$ is an integral domain.]

(a) Is the ideal $\langle 2x^0, x\rangle$ of $\mathbb{Q}[x]$ a principal ideal? Justify your answer. [**4**]

(b) Is $\mathbb{Q}[x]$ a unique factorisation domain? Justify your answer. [**4**]

(c) Is the factor ring $\mathbb{Q}[x]/\langle x^2 - 1\rangle$ an integral domain? Justify your answer. [**4**]

Question 7. (a) Let R be an integral domain. What is meant by a *Euclidean function* on R, and what does it mean for R to be a *Euclidean domain*? [**4**]

(b) Prove that if R is a Euclidean domain and I is an ideal of R then $I = aR$ for some $a \in R$ (where $aR = \{ar : r \in R\}$). [**10**]

End of Paper.