Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH4110 Mathematical Structures

Duration: 2 hours

Date and time: 7 May 2014, 14:30–16:30

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You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): P. J. Cameron

Question 1 (10 marks) (a) What is a prime number? [2]

(b) Prove that there are infinitely many prime numbers. [6]

(c) *Use your method of proof* to find a prime number different from 2, 3 and 5. [2]

Question 2 (10 marks) Let p be a prime number, and let $x_1, x_2, \ldots, x_p$ be natural numbers. Consider the statement

> If $x_1, x_2, \ldots, x_p$ are consecutive numbers, then at least one of them is divisible by p.

(a) Write down the *contrapositive* and the *converse* of this statement. [4]

(b) Is the original statement true? Give a proof or counterexample. [4]

(c) For each of the statements in your answer to (a), is that statement true or false? (Proofs not required.) [2]

Question 3 (10 marks) (a) Use *Euclid's algorithm* to find the greatest common divisor of 57 and 111. [4]

(b) Explain carefully why Euclid's algorithm, applied to any two natural numbers a and b, will terminate. (You are not required to show that it gives the right answer.) [6]

Question 4 (10 marks) (a) How many subsets of the set $\{1,2,3,4,5\}$ are there? [2]

(b) How many of these subsets contain three elements? [2]

(c) How many of the subsets in (a) contain the number 4? [3]

(d) How many of the subsets in (c) contain three elements? [3]

(You are not required to prove your assertions, but if you use a formula, you should state it clearly.)

Question 5 (10 marks) (a) Suppose that the relation R on the set $\mathbb{N}$ of natural numbers is defined by $x\,R\,y$ if and only if $x+y$ is even. Is R reflexive? Is it symmetric? Is it transitive? [6]

(b) You are given that the relation S on the set $\{1,2,3\}$ is an equivalence relation and has equivalence classes $\{1,2\}$ and $\{3\}$. Write down all the pairs (a,b) for which $a\,S\,b$ holds. [4]

Question 6 (10 marks) (a) What does it mean to say that a set X is *countably infinite*? [2]

(b) Prove that the set of all real numbers x between 0 and 1 is not countably infinite. [8]

Question 7 (10 marks) Let A be the set of all positive rational numbers, and let B and C be the subsets of A defined by

$$B = \{x \in A : x^2 < 2\}, \qquad C = \{x \in A : x^2 > 2\}.$$

(a) Show that $B \cap C = \emptyset$. [2]

(b) Why is $B \cup C = A$? (Give a brief explanation: detailed proof not required.) [4]

(c) Does B contain a greatest element? Give a brief explanation. [4]

Question 8 (10 marks) (a) Let $F : \mathbb{C} \to \mathbb{C}$ be the function defined by $F(z) = z^2$. Is F injective? Is it surjective? [3]

(b) Find a number $w \in \mathbb{C}$ satisfying $F(w) = 2\mathrm{i}$. [4]

(c) Let $G(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n over $\mathbb{C}$, where $n > 0$. Explain *briefly* why 0 is in the range of G. (You may use a theorem to show this but you should state the theorem.) [3]

Question 9 (10 marks) (a) Let p be a prime number. Explain why the binomial coefficients $\binom{p}{k}$, for $k = 1, 2, \ldots, p-1$, are all divisible by p. [2]

(b) State the *Binomial Theorem*. [2]

(c) Prove by induction that p divides $n^p - n$ for any natural number p. [6]

Question 10 (10 marks) (a) Find the flaw in the following proof: [8]

> **Theorem** If a and b are positive real numbers, then $\dfrac{a+b}{2} \geq \sqrt{ab}$.
>
> **Proof**
>
> $$\begin{array}{rrcl} & \dfrac{a+b}{2} & \geq & \sqrt{ab} \\ \Rightarrow & \dfrac{(a+b)^2}{4} & \geq & ab \\ \Rightarrow & (a+b)^2 & \geq & 4ab \\ \Rightarrow & a^2+2ab+b^2 & \geq & 4ab \\ \Rightarrow & a^2-2ab+b^2 & \geq & 0 \\ \Rightarrow & (a-b)^2 & \geq & 0 \end{array}$$
>
> which is true, because any number squared is ≥ 0. □

(b) How can it be fixed? [2]

End of Paper