Queen Mary University of London

Main Examination period 2019

MTH4104: Introduction to Algebra

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: A. Fink, F. Rincón

Question 1. [8 marks] Let $f, g \in \mathbb{R}[x]$ be polynomials, with $\deg g > 0$.

(a) The **division rule for polynomials** states that f can be divided by g to produce a quotient q and remainder r. Write down the two conclusions that the division rule states about q and r. **[2]**

(b) How do we tell, from q and r, whether g **divides** f? **[2]**

(c) Suppose that $\deg f = 8$, and $(x-1)^3$ divides f. What can be said about the multiplicity of $x = 1$ as a solution of $f(x) = 0$? **[4]**

Question 2. [14 marks]

(a) Define the following terms:

(i) **Cartesian product** of two sets; **[2]**

(ii) **relation** on a set X. **[2]**

(b) Write down a relation on the set $\{1, 2, 3\}$ which is reflexive and symmetric but not transitive. **[4]**

(c) Let S be the relation on the set $\mathbb{R} \setminus \{0\}$ defined by

$$xSy \text{ if and only if } y/x \in \mathbb{Q}.$$

Prove that S is an equivalence relation. **[6]**

Question 3. [22 marks]

(a) Define the **greatest common divisor** of two positive integers. **[2]**

(b) Use the extended Euclidean algorithm to compute the greatest common divisor d of 206 and 64, and to find integers x and y such that $206x + 64y = d$. **[16]**

(c) Write down another pair of integers (x', y') such that $206x' + 64y' = d$, different from the pair (x, y) you found in part (b). **[4]**

Question 4. [16 marks]

(a) Give the names of all axioms that must be satisfied in order for a set R with two operations $+$ and $\cdot$ to be a **ring**. [Do not write out what the axioms say.] **[6]**

(b) Name an example of a ring that is not a commutative ring. **[2]**

(c) Let R be a commutative ring. Prove that the identity $x^2 - y^2 = (x+y) \cdot (x-y)$ is true for all x and y in R. Name the axiom or proposition that you are using at each step of the proof. **[8]**

Question 5. [14 marks]

(a) Define what it means for an element of a ring with identity R to be a **unit**. **[2]**

(b) List all units in the ring $\mathbb{Z}_{12}$. **[4]**

(c) Is the matrix $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ a unit in the ring $\mathrm{M}_2(\mathbb{Z})$? Justify your answer. **[4]**

(d) Is the matrix $\begin{bmatrix} [2]_{12} & [1]_{12} \\ [-1]_{12} & [2]_{12} \end{bmatrix}$ a unit in the ring $\mathrm{M}_2(\mathbb{Z}_{12})$? Justify your answer. **[4]**

Question 6. [16 marks] Let g be the element

$$(1\;9\;11\;4\;6)(2\;5\;8)(3\;10\;7)$$

of S_{11}, written in cycle notation, and let h be the element

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 4 & 2 & 11 & 6 & 1 & 5 & 9 & 3 & 10 & 8 \end{pmatrix}$$

of S_{11}, written in two-line notation.

(a) Write g in two-line notation. **[3]**

(b) Find a permutation k such that $k \circ g = h$. Write k in two-line notation. **[8]**

(c) Define the **order** of a permutation. **[2]**

(d) Write down the order of g. **[3]**

Question 7. [10 marks]

(a) Define what it means for a set G with a binary operation $*$ to be a **group**, including the statements of every axiom you cite. **[4]**

(b) Let

$$S = \{a + b\mathrm{i} \in \mathbb{C} : a, b \in \mathbb{R}, a^2 + b^2 = 1\}$$

be the set of all complex numbers of modulus 1. Prove that S is a subgroup of the multiplicative group $\mathbb{C}^\times$. **[6]**

End of Paper.