Queen Mary
University of London

MTH4104: Introduction to Algebra

Duration: 2 hours

Date and time: 19 May 2016, from 10:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): A. Fink

Question 1.

(a) Let g be the element

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 2 & 10 & 5 & 4 & 9 & 1 & 7 & 3 & 8 \end{pmatrix}$$

of S_{10}, written in two-line notation. Write g in cycle notation. [**3**]

(b) What are the fixed points of g? [**2**]

(c) Find an element h of S_{10} such that

$$h \circ g = (1\ 8\ 10\ 2\ 7\ 3\ 6\ 4),$$

and write it in two-line notation. [**6**]

(d) Does S_{10} contain an element of order 30? If so, specify one. If not, explain why. [**4**]

Question 2.

(a) State the **Fundamental Theorem of Algebra**. [**3**]

(b) Find all solutions to the complex polynomial equation

$$z^4 + 8 - 8\sqrt{3}\mathrm{i} = 0,$$

and write them in standard form $a + b\mathrm{i}$. [**9**]

Question 3.

(a) Give a complete definition of what it means for a set G with an operation $\circ$ to be a **group**. [**3**]

(b) Write out the Cayley table for the multiplicative group $\mathbb{Z}_8^\times$. [**3**]

(c) Does the additive group $\mathbb{Z}_{30}$ have a subgroup of order 4? Specify one if so, or explain why if not. [**4**]

Question 4.

(a) State the names of the axioms that must hold of a set R with operations $+$ and $\cdot$ in order for R to be a **ring**. **[3]**

(b) Define what it means for an element of a ring with identity to be a **unit**. **[2]**

(c) Is $2 - 2t$ a unit in the ring $\mathbb{D}$ of pseudocomplex numbers? Justify your answer. **[4]**

(d) Let a be an element of a ring R with identity such that $a^n = 0$ for some natural number n. Prove that $1 - a$ is a unit in R. **[5]**

Question 5.

(a) Give complete definitions of the terms

(i) **Cartesian product** of two sets; **[2]**

(ii) **relation** on a set; **[2]**

(iii) **equivalence relation** on a set. **[3]**

(b) Write down examples of:

(i) a relation which is transitive but not reflexive; **[2]**

(ii) an equivalence relation on $\{1, 2, 3, 4\}$ with exactly three equivalence classes. **[2]**

(c) Let X and Z be any two sets, and $f : X \to Z$ any function. Prove that

$$\{(x, y) \in X^2 : f(x) = f(y)\}$$

is an equivalence relation on X. **[6]**

Question 6.

(a) Using the Euclidean algorithm, show that $\gcd(68, 183) = 1$. **[6]**

(b) Does $[68]_{183}$ have a multiplicative inverse in the ring $\mathbb{Z}_{183}$? Find it if so, or explain why if not. **[8]**

(c) Prove that $\mathbb{Z}_m$ is not a field if m is a composite number. (You may assume that $\mathbb{Z}_m$ is a ring, and that its operations are well-defined, but do not use other facts about $\mathbb{Z}_m$ without proof.) **[6]**

Question 7. Let T be the set of real numbers. Consider T as an algebraic structure with addition operation $\oplus$ and multiplication operation $\odot$ given by

$$x \oplus y = \min\{x, y\},$$
$$x \odot y = x + y - 2.$$

(a) Name the identity element in T for the operation $\odot$, and prove the inverse law for $\odot$. **[4]**

(b) Prove the distributive law in T. [Hint: consider two cases $x \leq y$, $x > y$.] **[3]**

(c) Prove that the set T with addition $\oplus$ and multiplication $\odot$ is not a ring. **[5]**

End of Paper.