Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH784P: Optimisation for Business Processes

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

The exam is closed-book, and **no outside notes are allowed**.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

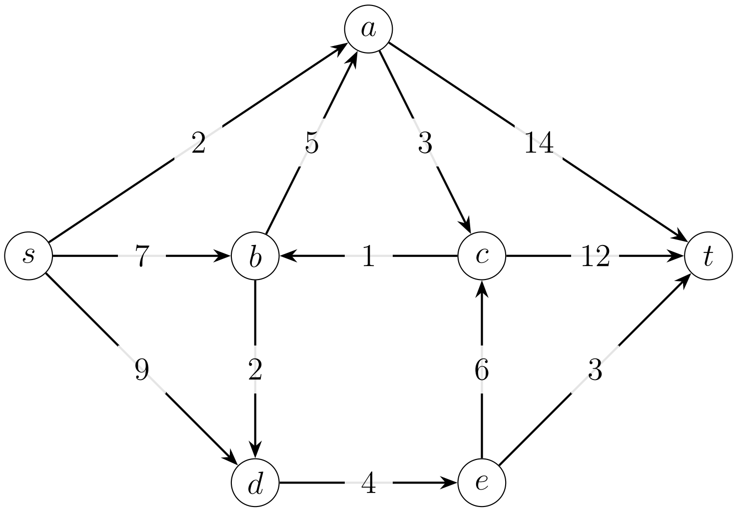
Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: M. Jerrum, V. Patel

Question 1 [25 marks]. Consider the following directed graph, where vertices are labeled by letters s, t and a–e, and edges by their (integer) lengths:



(a) Apply Dijkstra's algorithm to the graph, starting from vertex s. Describe clearly what the algorithm does in each step. What is the length of a shortest path from s to t? **[12]**

(b) Find a shortest path from s to t. Describe how you found the path. Is the shortest path unique? **[6]**

(c) We wish to reduce the length of the shortest path from s to t by reducing the length of a single edge. By how much would we have to reduce the length of edge ct in order to achieve this? **[3]**

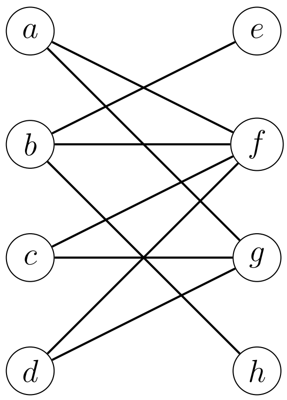
(d) Draw a directed graph G on 6 vertices, with distinguished vertices s and t, with the following property: in running Dijkstra's algorithm on G, the vertex label ('potential') of t is updated 5 times. If you cannot construct an example with 5 updates, try to achieve as many updates as possible. **[4]**

Question 2 [25 marks]. In this question, $G = (L, R, E)$ is a bipartite graph with vertex partition (L, R).

(a) Define the terms **matching**, **maximum matching** and **perfect matching** in G. [**3**]

(b) Describe how a maximum matching in G can be found by reduction to network flow. [**6**]

In the remainder of the question, take G to be the specific graph pictured below.



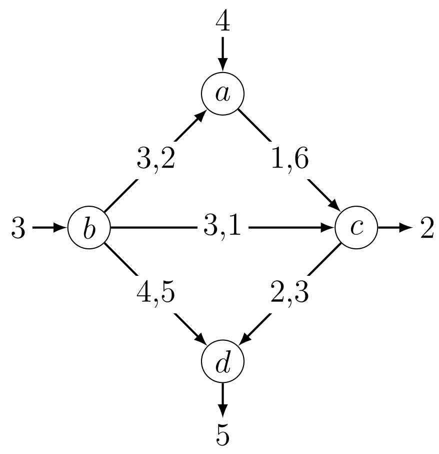
(c) Illustrate the construction you described in part (b) by applying it to the graph G. Call the resulting flow network $N(G)$. [**5**]

(d) Give the flow in $N(G)$ corresponding to the matching $\{ag, bf\}$ in G. [**3**]

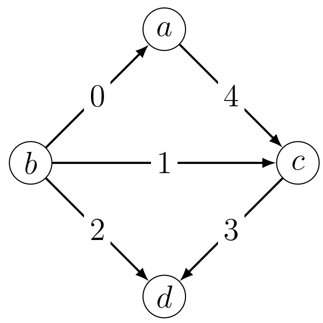
(e) Find an augmenting path for the flow given in part (d). Give the augmenting path, the new flow and the corresponding matching in G. [**4**]

(f) Let X be a subset of L. Explain briefly how the construction in part (b) could be modified to find a maximum matching that includes all the vertices in X. [**4**]

Question 3 [25 marks]. This question concerns a minimum-cost flow problem represented by the network below, where edges are labelled with their costs and capacities, in that order. Thus, edge ac has cost 1 and capacity 6. Also, vertices are labelled with their supplies/demands.



Consider the following feasible flow x in the network:



(a) Compute the cost of the given flow x. **[3]**

(b) Draw the residual graph corresponding to the flow x, indicating the costs and capacities of each of the edges. **[5]**

(c) Using the residual graph, find an augmenting cycle for the flow x, and write down the flow obtained by modifying the flow x using this augmenting cycle. What is the cost of the new flow? **[6]**

(d) Argue that the flow you found in part (c) is a minimum-cost flow. **[6]**

(e) Are there any other flows of minimum cost? Justify your answer. **[5]**

Question 4 [25 marks]. There are two kinds of animal feedstock, differing in nutritional content. In each tonne of feedstock, there are the following quantities of the nutrients 'starch', 'protein' and 'fibre' (in some appropriate units):

	starch	protein	fibre
Feedstock 1	3	1	1
Feedstock 2	1	1	2

Feedstock 1 costs £2000 per tonne and feedstock 2 costs £1000 per tonne.

(a) A farmer wishes to buy enough feedstock (of both sorts) so that in total there are at least 6 units of starch, 4 units of protein and 6 units of fibre, while at the same time minimising total cost. Formulate this optimisation problem as a linear program. [**8**]

(b) Solve the linear program from part (a) graphically. How many tonnes of each kind of feedstock should the farmer buy? What is the total cost? [**8**]

(c) As a first step towards solving the problem algebraically, transform your linear program from part (a) to standard equality form. [**5**]

(d) Referring to your sketch from part (b), write down all basic feasible solutions to the linear program from part (c). [**4**]

End of Paper.