Queen Mary University of London

Main Examination period 2022 – May/June – Semester B

MTH6105: Algorithmic Graph Theory

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

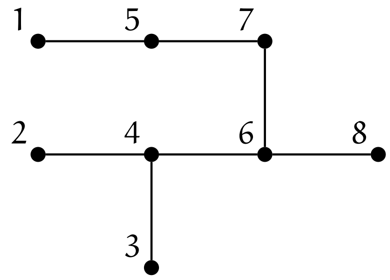
Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: F. Fischer, J. Ward

In this paper, $\mathsf{V}(\mathsf{G})$ denotes the set of vertices of a graph or digraph G, $\mathsf{E}(\mathsf{G})$ the set of edges of a graph G, and $\mathsf{A}(\mathsf{G})$ the set of arcs of a digraph G. You may use any result from lecture notes and exercises without proving it, but you must state clearly which result you use.

Question 1 [24 marks].

(a) Give the Prüfer code of the following tree. [**4**]



(b) Give the degree sequence of the tree with Prüfer code $6, 4, 3, 2, 1, 6, 4, 6$. Show your working. [**6**]

(c) Draw a tree with degree sequence $3, 3, 2, 2, 1, 1, 1, 1$. [**2**]

(d) Show that any connected graph with degree sequence $3, 3, 2, 2, 1, 1, 1, 1$ must be a tree. [**6**]

(e) Give an efficient algorithm that, for any sequence $d_1, d_2, \ldots, d_n$ that is the degree sequence of a tree, constructs a particular tree with that degree sequence. Explain briefly why the algorithm is correct and efficient. [**6**]

Question 2 [22 marks]. Consider the network (G, w) with

$$V(G) = \{u, a, b, c, d, e, f, g\},$$
$$E(G) = \{ua, ub, uc, ac, ad, af, bc, bg, cd, ce, df, eg, fg\}, \text{ and}$$

$$w(ua) = 6, \quad w(ub) = 3, \quad w(uc) = 1, \quad w(ac) = 4, \quad w(ad) = 2,$$
$$w(af) = 1, \quad w(bc) = 1, \quad w(bg) = 5, \quad w(cd) = 1, \quad w(ce) = 1,$$
$$w(df) = 4, \quad w(eg) = 6, \quad w(fg) = 1.$$

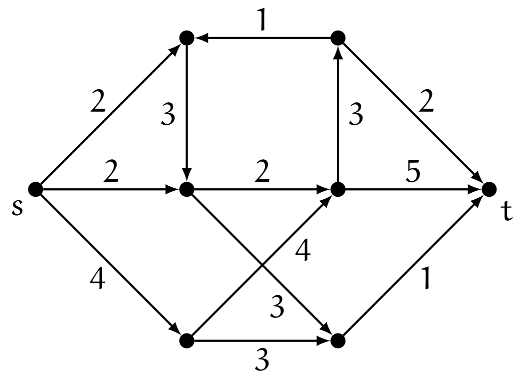
(a) Draw G. **[2]**

(b) Apply Dijkstra's algorithm to (G, w) starting from vertex u. Give $V(T)$ and $E(T)$ after each iteration of the algorithm. **[10]**

(c) For each $v \in V(G)$, give the length of a shortest u–v-path in (G, w). Justify your answer. **[4]**

(d) Show that the tree T obtained in Part (b) is the unique minimum spanning tree of (G, w). **[6]**

Question 3 [26 marks]. Consider the directed network (D, c) given by the following drawing, where each arc $e \in A(D)$ is labelled by its capacity $c(e)$ and two vertices s and t have been identified.



(a) Use the Ford-Fulkerson algorithm to find a maximum s—t-flow of (D, c). Draw the residual network after each iteration of the algorithm, and give the size of the maximum flow. **[14]**

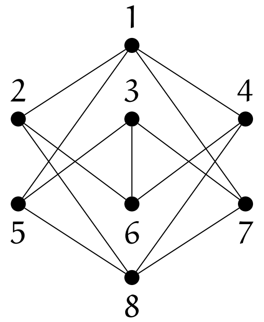
(b) Use a cut to show that the flow you have found is indeed a maximum s—t-flow of (D, c). **[6]**

(c) If the capacity of exactly one of the arcs with capacity 4 was increased to 5, would this affect the size of a maximum flow? Justify your answer. **[6]**

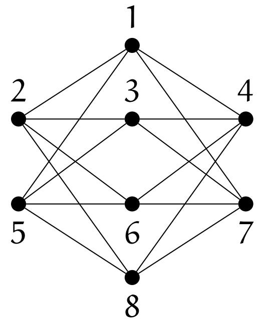
Question 4 [28 marks].

(a) For each of the following graphs, state whether the graph is bipartite or not. Justify your answers. **[4]**

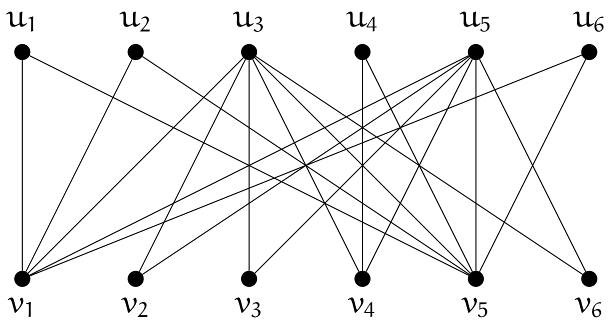
(i)



(ii)



Consider the bipartite graph G given by the following drawing, where each vertex is labelled with its name.



(b) Show that $M = \{u_2v_5, u_3v_4, u_5v_2, u_6v_1\}$ is a matching of G. **[2]**

(c) Find a maximum matching of G. Show your working. **[10]**

(d) Use Hall's theorem to show that the matching you have found is indeed a maximum matching. **[6]**

Call a graph H **regular** if there exists $k \geq 1$ such that $d_H(v) = k$ for all $v \in V(H)$.

(e) Does there exist a regular graph H with $V(H) = V(G)$ and $E(H) \subseteq E(G)$? Justify your answer. **[6]**

End of Paper.