Queen Mary University of London

Main Examination period 2020 – May/June – Semester B
Online Alternative Assessments

MTH5114: Linear Programming and Games

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment, you may use books, notes, and the internet. You may use calculators and computers, but you should show your working for any calculations you do. You must not seek or obtain help from anyone else.

At the start of your work, please **copy out and sign** the following declaration:

> I declare that my submission is entirely my own, and I have not sought or obtained help from anyone else.

All work should be **handwritten**, and should **include your student number**.

You have **24 hours** in which to complete and submit this assessment. When you have finished your work:

- scan your work, convert it to a **single PDF file** and upload this using the upload tool on the QMplus page for the module;
- e-mail a copy to maths@qmul.ac.uk with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are not expected to spend a long time working on this assessment. We expect you to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the assessment period, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: J. Ward, M. Jerrum

Question 1 [24 marks].

(a) Rewrite the following linear program in **standard inequality form**:

$$\begin{aligned}
\text{minimise} \quad & 7x_1 - x_2 + 5x_3 - 10x_4 \\
\text{subject to} \quad & -11x_1 - 12x_2 - x_3 + x_4 \geq 1, \\
& x_1 - 4x_2 - 8x_3 = 12, \\
& 2x_1 + 6x_2 + 3x_4 \leq 7, \\
& x_1, x_2 \geq 0, \\
& x_3 \leq 0, \\
& x_4 \text{ unrestricted}
\end{aligned}$$

[**6**]

(b) Consider the following linear program in standard equation form:

$$\begin{aligned}
\text{maximise} \quad & x_1 + 2x_2 - 3x_3 + 7x_5 \\
\text{subject to} \quad & x_1 + 2x_2 + 2x_3 + x_4 = 3, \\
& x_1 + 2x_2 + 7x_3 + x_5 = 3, \\
& 2x_1 + 4x_2 + 7x_3 + x_6 = 6, \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}$$

For each of the following values of $\mathbf{x}^\mathsf{T} = (x_1, x_2, x_3, x_4, x_5, x_6)$ say whether or not this value is a **basic feasible solution** of this linear program. Justify your answers. [**6**]

(i) $\mathbf{x}^\mathsf{T} = (1, 1, 0, 0, 0, 0)$

(ii) $\mathbf{x}^\mathsf{T} = (1, 0, 0, 2, 2, 4)$

(iii) $\mathbf{x}^\mathsf{T} = (0, 0, 0, 3, 3, 6)$

(c) Which, if any, of (i), (ii), and (iii) from question 1(b) are **extreme point solutions** to the given linear program? Justify your answer. You may use, without proving, any theorems discussed in the lectures. [**4**]

(d) Consider an arbitrary linear program in standard equation form:

$$\begin{aligned}
\text{maximise} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

Suppose that $\mathbf{x}$ is an optimal solution to this linear program. Show that if $\mathbf{x}$ is **not** an extreme point solution then we can express $\mathbf{x}$ as $\mathbf{x} = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$ where $\lambda \in (0, 1)$ and $\mathbf{y}$ and $\mathbf{z}$ are two different optimal solutions of this program. [**8**]

Question 2 [17 marks]. A power company operates three power generation plants. One is a wind plant, and the other two consume a combination of Fuel 1 and Fuel 2, emitting carbon dioxide in the process. In addition, all three plants require maintenance. The amount of fuel consumed (in Mg), maintenance required (in person-hours), carbon dioxide (CO_2) emitted (in Mg), and power generated (in MWh) per day of operation is as follows:

Plant	Maintenance Required	Fuel 1 Required	Fuel 2 Required	CO_2 Emitted	Power Produced
1	20	0	0	0	20
2	13	10	15	12	32
3	18	30	40	29	40

Each MWh of power can be sold at £121 and there is no limit on the amount that can be sold. Over its next planning period, the company has 230 person-hours for maintenance, 75 Mg of Fuel 1, and 90 Mg of Fuel 2 available.

(a) Due to environmental regulations, they cannot emit more than 200Mg of CO_2 in this period. The company wants to know how to operate its plants to generate as much revenue as possible (you may assume that there is no limit on the number of days a plant can operate in this period). Give a linear program that models this problem and state what each of your variables is meant to represent. You do not need to solve this program. [**11**]

(b) Suppose now that the company can emit more than 200Mg of CO_2, but now loses £55 of revenue for each Mg emitted after the first 200Mg because it must purchase "CO_2 credits". The other resource constraints remain as stated. The company now wants to know how to operate its plants to generate as much revenue as possible (you may assume that there is no limit on the number of days that a plant can operate in this period). Give a linear program that models this problem. You do not need to solve this program. [**6**]

Question 3 [23 marks].

(a) Suppose that at some point when carrying out the standard simplex algorithm on a linear program in standard equation form, you have the following current tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	1	0	1	-2	0	4
x_3	1	-2	1	0	-2	0	6
x_6	3	2	0	0	-3	1	12
$-z$	-6	1	0	0	-1	0	-9

(i) List the values of all variables in the basic feasible solution corresponding to this tableau. **[2]**

(ii) What is the objective value of this basic feasible solution? **[2]**

(iii) What is the entering variable and what is the leaving variable for the next pivot operation? **[2]**

(iv) Finish running the simplex algorithm for this linear program. For each pivot step, state the entering and leaving variable for this step together with the next tableau. If the linear program has an optimal solution, give it. If not, explain why. **[8]**

(b) Suppose we wanted to solve the following linear program with the 2-phase simplex algorithm:

$$\begin{aligned} \text{maximise} \quad & x_1 + 7x_2 - x_3 \\ \text{subject to} \quad & x_1 - x_2 \geq 3, \\ & -3x_1 + x_2 - 2x_3 = 2, \\ & x_1 + 2x_2 + x_3 \leq 4, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(i) Give the initial tableau for phase 1 of the 2-phase simplex algorithm. Make sure your tableau is in a valid form. **[6]**

(ii) Based on the initial tableau, can you determine whether this program is feasible or not? Justify your answer. **[3]**

Question 4 [23 marks].

(a) Give the dual of the following linear program:

$$\begin{aligned} \text{maximise} \quad & 70x_1 + 80x_2 + 90x_3 \\ \text{subject to} \quad & x_1 - 2x_2 + 3x_3 \leq 5, \\ & 2x_1 + 4x_2 + 6x_3 \leq 10, \\ & 3x_1 - 8x_2 - x_3 = 20, \\ & x_2, x_3 \geq 0, \\ & x_1 \text{ unrestricted} \end{aligned}$$

[**8**]

(b) Consider the following linear program in standard inequality form:

$$\begin{aligned} \text{maximise} \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Suppose that this program is unbounded. Show that the dual of this program must be infeasible. You may use any theorems and definitions from the lectures. [**8**]

(c) Consider the following linear program:

$$\begin{aligned} \text{maximise} \quad & 4x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & x_1 - x_2 + 3x_3 \leq 1, \\ & 5x_1 + x_2 + 8x_3 \leq 60, \\ & -x_1 + 2x_2 - 5x_3 \leq 3, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Show that $x_1 = 0$, $x_2 = 14$, $x_3 = 5$ is an optimal solution to this program, and give an optimal solution $\mathbf{y}$ to the dual of this program. Show your working. [**7**]

Question 5 [13 marks].

(a) Consider the following 2-player zero-sum game. Each player separately chooses a number from the set $\{1, 2, 3\}$. Both players then reveal their numbers. If the numbers match, the row player must pay £3 to the column player, otherwise, the player with the lower number must pay £1 to the player with the higher number.

(i) Give the payoff matrix for this game from the perspective of the row player. Also give the security level for each of the player's strategies. [**5**]

(ii) Does this game have a pure Nash equilibrium? If so, give **all** pure Nash equilibria for the game. If not, explain why. [**3**]

(iii) Formulate a linear program that finds the row player's best mixed strategy in this game (you do not need to solve this program). [**5**]

End of Paper.