Queen Mary
University of London

Main Examination period 2020 – May – Semester B

MTH4115 / MTH4215: Vectors & Matrices

Duration: 2 hours

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Examiners: O. Jenkinson, R. Johnson

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Question 1 [20 marks]. Let A, B, C be points in 3-space with respective position vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$. Determine:

(a) The length of the vector $\mathbf{a}+\mathbf{b}+\mathbf{c}$; **[3]**

(b) A unit vector in the direction of $\mathbf{a}$; **[3]**

(c) $\mathbf{a} \cdot \mathbf{b}$; **[3]**

(d) $\mathbf{a} \times \mathbf{b}$; **[3]**

(e) A vector equation for the line through A and B; **[4]**

(f) A Cartesian equation for the plane containing A, B and C. **[4]**

Solutions [All parts are routine calculations]:

(a) $\mathbf{a}+\mathbf{b}+\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix}$, which has length $\sqrt{1+0+49} = \sqrt{50} = 5\sqrt{2}$.

(b) $\mathbf{a}$ has length $\sqrt{2}$, so the unit vector in the direction of $\mathbf{a}$ is $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$.

(c) $\mathbf{a} \cdot \mathbf{b} = -1+0+3 = 2$.

(d) $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -3 \\ -4 \\ 3 \end{pmatrix}$.

(e) $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a})$, $\lambda \in \mathbb{R}$, is such an equation, which in this case becomes
$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

(f) $\mathbf{n} = (\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ 6 \end{pmatrix}$ is orthogonal to this plane, and since A is contained in the plane then an equation for it is $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = 18$, which in Cartesian form is $12x + 4y + 6z = 18$, or alternatively $6x + 2y + 3z = 9$.

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Question 2 [20 marks]. Let Π be the plane with equation $2x+y+z=1$, let l be the line with equations $x=y=z$, and let Q be the point with position vector $\mathbf{q}=\begin{pmatrix}0\\1\\2\end{pmatrix}$.

(a) Determine the distance between the point Q and the plane Π. **[4]**

(b) Determine the coordinates of the point on Π that is closest to Q. **[4]**

(c) Determine the distance between the point Q and the line l. **[4]**

(d) Determine the point of intersection of the line l and the plane Π. **[4]**

(e) If l' is the line in the direction orthogonal to Π and passing through Q, then determine the distance between l and l'. **[4]**

Solutions: [All parts are fairly routine use of formulae from lectures and practiced on exercise sheets; part (e) should be slightly more challenging]

(a) The vector $\mathbf{n}=\begin{pmatrix}2\\1\\1\end{pmatrix}$ is orthogonal to Π, so this distance (using the formula derived in lectures) is $|\mathbf{q}\cdot\mathbf{n}-1|/|\mathbf{n}|=2/\sqrt{6}=\sqrt{6}/3$.

(b) Using the formula from lectures, this closest point has position vector

$$\mathbf{q}-\left(\frac{\mathbf{q}\cdot\mathbf{n}-1}{|\mathbf{n}|^2}\right)\mathbf{n}=\begin{pmatrix}0\\1\\2\end{pmatrix}-(1/3)\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}-2/3\\2/3\\5/3\end{pmatrix},$$

so its coordinates are $(-2/3,2/3,5/3)$.

(c) The line l has vector equation $\mathbf{r}=\lambda\mathbf{u}$ where $\mathbf{u}=\begin{pmatrix}1\\1\\1\end{pmatrix}$, so the required distance is $|\mathbf{u}\times\mathbf{q}|/|\mathbf{u}|$ (a formula from lectures). Now $\mathbf{u}\times\mathbf{q}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$, so $|\mathbf{u}\times\mathbf{q}|=\sqrt{6}$, and $|u|=\sqrt{3}$, therefore the required distance is $|\mathbf{u}\times\mathbf{q}|/|\mathbf{u}|=\sqrt{6}/\sqrt{3}=\sqrt{2}$.

(d) The point of intersection has coordinates $(1/4,1/4,1/4)$.

(e) The line l' has vector equation $\mathbf{r}=\mathbf{q}+\lambda\mathbf{n}$, so by a formula from lectures the required distance is $|\mathbf{q}\cdot(\mathbf{u}\times\mathbf{n})|/|\mathbf{u}\times\mathbf{n}|$. Now $\mathbf{u}\times\mathbf{n}=\begin{pmatrix}0\\1\\-1\end{pmatrix}$, so $|\mathbf{u}\times\mathbf{n}|=\sqrt{2}$, and $\mathbf{q}\cdot(\mathbf{u}\times\mathbf{n})=1-2=-1$, so the required distance is $|\mathbf{q}\cdot(\mathbf{u}\times\mathbf{n})|/|\mathbf{u}\times\mathbf{n}|=1/\sqrt{2}$.

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Question 3 [10 marks].
Consider the linear system

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & + & x_3 & + & 4x_4 & = & 1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & + & x_4 & = & 0 \\ 2x_1 & + & 3x_2 & - & 2x_3 & + & 7x_4 & = & 2 \\ 2x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 1\,. \end{array}$$

(a) The augmented matrix of this system is: **[3]**

(b) After bringing the augmented matrix to row echelon form, the leading variables of the reduced system are: **[3]**

(c) The solution set of the system is: **[4]**

Solutions [Similar to examples seen in lectures and on exercise sheets]:

(i) The augmented matrix of the system is

$$\left(\begin{array}{cccc|c} 2 & 3 & 1 & 4 & 1 \\ 2 & 3 & 4 & 1 & 0 \\ 2 & 3 & -2 & 7 & 2 \\ 2 & 3 & 4 & -2 & 1 \end{array}\right).$$

(ii) The leading variables are x_1, x_3 and x_4

(iii) We see that $x_4 = -1/3$, and $x_3 = x_4 - 1/3 = -2/3$, and $x_1 = -(3/2)x_2 - (1/2)x_3 - 2x_4 + 1/2 = -(3/2)x_2 + 3/2$, so the solution set is

$$\left\{ (-\frac{3}{2}\alpha + \frac{3}{2}, \alpha, -\frac{2}{3}, -\frac{1}{3}) : \alpha \in \mathbb{R} \right\}.$$

End of Paper.

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