Queen Mary
University of London

Main Examination period 2017

MTH5112 : Linear Algebra I

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: C. Busuioc

Question 1.

(a) Let A be a 3×3 matrix with real entries such that $\det(A) = 5$.

(i) Suppose the matrix B is obtained from A by interchanging the first two rows and multiplying the third row by 3. Find $\det(B)$. **[4]**

(ii) Explain why A is invertible and show that $\det(A^{-1}) = \frac{1}{5}$. **[4]**

(iii) Find $\det(2A^T A^{-1})$. **[6]**

(b) Let $A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 4 & 0 \\ 5 & 0 & -1 & 2 \end{pmatrix}$. Consider the products AB and $(BA)^T$. In each case, either compute the given product, or explain why the operation cannot be performed. **[4]**

Question 2. Consider the linear system

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 0 \\ 2x_1 & - & 3x_2 & + & 4x_3 & - & 3x_4 & = & 0 \\ -x_1 & + & x_2 & - & 3x_3 & + & 2x_4 & = & 0 \end{array} .$$

(a) Write down the augmented matrix of the system. **[2]**

(b) Bring the augmented matrix to **reduced row echelon form.** Indicate the elementary row operations used at each step. **[4]**

(c) Identify the leading and the free variables, and write down the solution set of the system. **[4]**

Question 3.

(a) Give an example of 5 linearly independent vectors in $\mathbb{R}^4$ or explain why it is impossible to do so. **[4]**

(b) Give an example of 3 vectors that span $\mathbb{R}^2$ or explain why it is impossible to do so. **[4]**

(c) Let $H = \left\{ A \in \mathbb{R}^{2\times 2} \,\middle|\, A \text{ is diagonal} \right\}$.

(i) Show that H is a subspace of $\mathbb{R}^{2\times 2}$. **[6]**

(ii) Write down a basis for H and determine the dimension of H. A proof that your answer does indeed provide a basis for H is not required. **[4]**

Question 4. Let $u_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$ and let $\mathscr{B} = \{u_1, u_2, u_3\}$.

(a) Show that $\mathscr{B}$ is an orthogonal basis of $\mathbb{R}^3$. **[8]**

(b) Determine the transition matrix from $\mathscr{B}$ to the standard basis of $\mathbb{R}^3$. **[4]**

(c) Suppose the $\mathscr{B}$-coordinate vector of $u \in \mathbb{R}^3$ is

$$[u]_{\mathscr{B}} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}.$$

Find u in standard basis. **[4]**

Question 5.

(a) Suppose $A \in \mathbb{R}^{m\times n}$. Give the definition of $\operatorname{rank}(A)$ and $\operatorname{nul}(A)$. **[2]**

(b) State the Rank–Nullity Theorem. **[2]**

(c) If the column space of a 9×4 matrix is 3-dimensional, what is the dimension of the null space? Justify your answer. **[4]**

(d) Is it true that if the determinant of a 10×10 matrix A is 10, then the rank of A is 10? Justify your answer. **[4]**

Question 6. Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{pmatrix}.$$

(a) Find the eigenvalues of A and their corresponding eigenspaces. **[10]**

(b) Explain why A is diagonalisable. **[2]**

(c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. **[4]**

Question 7. Consider the least squares problem $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ -1 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}.$$

(a) Write down the corresponding normal equations. **[6]**

(b) Determine the set of least squares solutions to the problem. **[4]**

End of Paper.