Queen Mary University of London

B. Sc. Examination by course unit 2015

MTH5112: Linear Algebra I

Duration: 2 hours

Date and time: 1 May 2015, 10:00 - 12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): C. Beck

Question 1.

(a) Which of the following matrices (if any) are in row echelon form?

$$\text{(i)} \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

[**2**]

(b) Consider the linear system

$$\begin{array}{rcrcrcrcr} 3x_1 & - & 6x_2 & + & 3x_3 & + & 9x_4 & = & 3 \\ 2x_1 & - & 3x_2 & + & 3x_3 & + & 4x_4 & = & 4 \\ -3x_1 & + & 7x_2 & - & 2x_3 & - & 10x_4 & = & -1 \end{array}$$

(i) Write down the augmented matrix of the system. [**2**]

(ii) Transform the augmented matrix to row echelon form. Indicate which elementary row operation you use at each step. [**3**]

(iii) Identify the leading and the free variables, and write down the solution set of the system. [**3**]

Question 2. Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

(a) Calculate A^2 and A^3. [**4**]

(b) Define what it means for a matrix M to be invertible and what is meant by the inverse of M. [**2**]

(c) Use your answer to (a) to show that A is invertible. [**4**]

Question 3. Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 3 \\ 9 & 0 & 1 & 8 \\ -8 & 2 & 4 & 5 \\ 3 & 0 & 0 & 5 \end{pmatrix}.$$

(a) Calculate $\det(A)$ and decide whether A is invertible or not. [**4**]

(b) Using (a) evaluate $\det(A^8)$ and $\det(3A)$. In each case, briefly explain which property of determinants you are using. [**4**]

(c) Find $\det(B)$, where B is the matrix obtained from A by subtracting 17 times column 1 from column 4. [**2**]

Question 4.

(a) Which of the following statements (if any) are true? Give detailed reasons for your answers.

(i) $H_1 = \left\{ (r,s)^T \,\middle|\, r,s \in \mathbb{R} \text{ and } r - 3s = 0 \right\}$ is a subspace of $\mathbb{R}^2$.

(ii) $H_2 = \left\{ A \in \mathbb{R}^{n\times n} \,\middle|\, A + A^T = 2I \right\}$ is a subspace of $\mathbb{R}^{n\times n}$ for any $n \in \mathbb{N}$.

[**4**]

(b) Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be vectors in a vector space. What does it mean for $\mathbf{v}_1, \dots, \mathbf{v}_n$ to be *linearly independent*? [**2**]

(c) Show that $\left\{(7,0,0,0)^T, (2,10,0,0)^T, (4,5,3,0)^T, (7,0,9,1)^T\right\}$ is a basis of $\mathbb{R}^4$. [**4**]

Question 5.

(a) Define what is a *linear transformation.* [**2**]

(b) Let the set P_n be the set of all polynomials $\mathbf{p}(t)$ of degree n or less, and let $L : P_{10} \to P_2$ be the mapping given by

$$(L(\mathbf{p}))(t) = (1 + 2t^2)\mathbf{p}(3)\,.$$

Is L a linear transformation? Prove your answer. [**4**]

(c) Let the mapping $M : \mathbb{R}^2 \to \mathbb{R}^1$ be given by $M(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$, where $\mathbf{x} = (x_1, x_2)^T$. Is M a linear transformation? Prove your answer. [**4**]

Question 6. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 5 \\ 8 \\ -6 \\ 6 \end{pmatrix}.$$

(a) Which of the following sets (if any) are orthogonal? Give reasons for your answers.

(i) $\{\mathbf{u}_1, \mathbf{u}_2\}$.

(ii) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{y}\}$.

[**4**]

(b) Let $H = \operatorname{Span}(\mathbf{u}_1, \mathbf{u}_2)$. Write $\mathbf{y}$ as a sum of a vector in H and a vector in $H^\perp$. [**4**]

(c) Determine an orthogonal basis of $\operatorname{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{y})$. [**4**]

Question 7. Let

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix}.$$

(a) Is the linear system $A\mathbf{x} = \mathbf{b}$ consistent? Give reasons. [**2**]

(b) Determine $\mathbf{x} \in \mathbb{R}^2$ that makes $\|A\mathbf{x} - \mathbf{b}\|$ as small as possible. [**5**]

Question 8. Let

$$A = \begin{pmatrix} 3 & 6 & 0 \\ -1 & -2 & 0 \\ 1 & 4 & -2 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -9 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A and find the corresponding eigenvalues. [**4**]

(b) Find the characteristic polynomial of A and factorise it. [**4**]

(c) Determine all eigenvalues of A and find bases for the corresponding eigenspaces. [**4**]

(d) Is A diagonalisable? If it is, write down a matrix that diagonalises A. [**2**]

Question 9.

(a) Define what is a *symmetric matrix*. [**2**]

(b) Define what is an *orthogonal matrix*. [**2**]

(c) State the *Spectral Theorem for symmetric matrices*. [**4**]

Question 10.

(a) Define what is the *nullspace* of a matrix A. [**2**]

(b) Explain what it means that two matrices A and B *commute*. [**2**]

(c) Let A be a square matrix that commutes with its transpose. Show that the nullspaces of A and A^T coincide. [Hint: show that $\|A\mathbf{x}\|^2 = \|A^T\mathbf{x}\|^2$.] [**5**]

End of Paper.