Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH5112 Linear Algebra I

Duration: 2 hours

Date and time: 9 May 2014, 14:30–16:30

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You should attempt all questions. Marks awarded are shown next to the questions.

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Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): R.A. Wilson

Question 1

(a) For each of the following matrices state whether or not it is in row echelon form. Also state whether or not it is in reduced row echelon form.

$$\text{(i)}\ \begin{pmatrix} 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{(ii)}\ \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

[4]

(b) Compute the determinant of the matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ -1 & 2 & -2 \end{pmatrix}.$$

Is this matrix invertible? If so, find its inverse; if not, prove it. [6]

(c) Hence or otherwise solve the linear system

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 1 \\ 2x_1 & & & + & x_3 & = & 2 \\ -x_1 & + & 2x_2 & - & 2x_3 & = & -3 \end{array}$$

(Show your working in full.) [4]

Question 2

(a) Let

$$A = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}.$$

For each of the products A^2, AB, BA, B^2, state whether or not it exists; if it exists then evaluate it. [4]

(b) Suppose that B is a square matrix and $B^2 = O$. Prove that $I + B$ is invertible, and find its inverse. [4]

Question 3 For each of the following statements about $n \times n$ matrices A and B, state whether it is true or false, and provide a proof or counterexample as appropriate.

(a) $\det(A + B) = \det A + \det B$; [4]

(b) $\det(A + B^T) = \det(B + A^T)$; [4]

(c) $\det(BA) = \det(AB)$; [4]

(d) $\det(\lambda A) = \lambda \det A$, where $\lambda \in \mathbb{R}$. [4]

Question 4

(a) Let V be a vector space, and let $S = \{v_1, \ldots, v_k\}$ be a set of k distinct non-zero vectors in V. Explain what is meant by the statement that S is a *spanning set* for V. Explain what is meant by the statement that S is a *linearly dependent set.* [6]

(b) Let $V = \mathbb{R}^4$, and let

$$S = \{(0, 2, 1, -3)^T, (1, 1, 1, 2)^T, (3, 0, -1, -1)^T, (-1, 4, 4, 2)^T\}.$$

Either determine whether or not S is a spanning set for V,

or determine whether or not S is a linearly dependent set. [4]

Question 5

Let P_2 denote the set of polynomials of degree at most 2, that is

$$P_2 = \{\mathbf{p} \mid \mathbf{p}(x) = a_2x^2 + a_1x + a_0 \text{ for some } a_0, a_1, a_2 \in \mathbb{R}\}.$$

Let $D : P_2 \to P_2$ be the mapping given by $D(\mathbf{p}) = \mathbf{q}$, where $\mathbf{p}'$ denotes the derivative of $\mathbf{p}$, and

$$\mathbf{q}(x) = x^2\mathbf{p}(0) + \mathbf{p}'(x).$$

(a) Show that D is a linear transformation. [4]

(b) Is D surjective? Justify your answer. [4]

(c) Is D injective? Justify your answer. [4]

Question 6

(a) Let x and y be vectors in $\mathbb{R}^n$. Define the scalar product (or dot product) $x.y$. [2]

(b) Let H be a subspace of $\mathbb{R}^n$. Define the *orthogonal complement* $H^\perp$ of H. [2]

(c) Explain why $\dim H + \dim H^\perp = n$. [4]

(d) Suppose $n = 4$ and

$$H = \{(x, y, z, t)^T \mid x + 2y - 3z = 0, 3y + 2z - t = 0\}.$$

Compute $\dim H$, $\dim H^\perp$, and a basis for $H^\perp$. [6]

Question 7

Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} -1 \\ 4 \\ 1 \\ -3 \end{pmatrix}.$$

Let $H = \text{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

(a) Apply the Gram–Schmidt process to the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to obtain an orthogonal basis for H. [6]

(b) Let $\mathbf{y} = (2, 0, 3, -1)^T$. Express $\mathbf{y}$ as the sum of a vector in H and a vector in $H^\perp$, and hence find the closest point to $\mathbf{y}$ in H. [6]

Question 8 Let

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 3 & 2 \\ 2 & 4 & 0 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

(a) Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A and find the corresponding eigenvalues. [4]

(b) Find the characteristic polynomial of A. [4]

(c) Determine all eigenvalues of A and find bases for the corresponding eigenspaces. [4]

(d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. [2]

End of Paper