Queen Mary University of London

Main Examination period 2022 – May/June – Semester B

MTH6155: Financial Mathematics II

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final.**

Examiners: I. Goldsheid, V. Fain

1. The following convention is used in this paper. If $Y(t)$ is a random process then Y_t may be used to describe the same process; a similar convention applies to any other random process. In particular, throughout this paper both $W(t)$ and W_t denote the standard Wiener process.
2. $\tilde{\mathbb{E}}$ denotes the expectation over a risk-neutral probability.
3. Time involved in calculations should be expressed in years. E. g., 3 months should be converted into 0.25 years.
4. The precision of calculations should be to 3 decimal places.
5. You may use without proof the following equalities. If $X \sim \mathcal{N}(0, \sigma^2)$ then $\mathbb{E}(e^X) = e^{\frac{\sigma^2}{2}}$. In particular, $\mathbb{E}(e^{bW_t}) = e^{\frac{b^2}{2}t}$, where b is any real number.

Question 1 [24 marks]. This question is about the Wiener process and the geometric Brownian motion.

(a) The random process $Y(t),\ t \geqslant 0$, is defined by

$$Y(t) = W(t)^2,$$

where $W(t),\ t \geqslant 0$, is the standard Wiener process.

(i) Compute $\mathrm{Cov}(Y_t, Y_s)$ and derive from this result the expression for the variance of Y_t. **[8]**

Hint. Note that
$W_t^2 = [(W_t - W_s) + W_s]^2 = (W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2$, where $(W_t - W_s)$ and $W(s)$ are independence random variables.

(ii) Compute the expectation of the product $(Y_t - Y_s)Y_s$, where $t \geqslant s \geqslant 0$. **[5]**

(iii) Does the process $Y(t)$ have independent increments? **[3]**

(b) Consider the geometric Brownian motion of the form $S(t) = e^{\sigma W(t)}$. Compute the expectation of the product $S(t)S(2t)S(3t)$. **[8]**

Question 2 [9 marks]. This question is about the Arbitrage Theorem.

(a) State the Arbitrage Theorem. **[3]**

(b) You are playing the following game. A coin is tossed. The two possible outcomes are **heads** and **tails**. We denote them by 1 and 2 respectively.

You can bet on any outcome. The conditions of the bets are as follows.

If you bet £1 on outcome 1 and the outcome is 1, then you get back your pound and a reward of £u, where $u > 0$. But if the outcome is 2 then you lose your pound.

If you bet £1 on outcome 2 and the outcome is 2 then you get back £0.5 and a reward of £$3u$. But if the outcome is 1 then you lose your pound.

For what value of u will this game be arbitrage-free?

Justify your answer. **[6]**

Question 3 [19 marks]. The price of a share follows the geometric Brownian motion with parameters $\mu = 0.1$ and $\sigma = 0.2$. Presently, the share's price is £33. The continuously compounded interest rate is 5%.

(a) A European put option on this share has expiration time T and the strike price K. Using the Black-Scholes formula for the price C of a Call(K, T), write down the formula for the price P of this put option. [**5**]

Hint Use the call-put parity formula.

(b) Suppose now that the put option has the expiration time $T = 1.5$ years and the strike price $K =$ £35. A trader who sells this option at time $t = 0$ for £P has to design a hedging strategy which would allow him to meet his financial obligation in 1.5 year's time. The hedging portfolio should consist of underlying shares and of money deposited in the bank.

(i) State the formulae allowing one to compute the number of shares in the portfolio and the capital deposited in the bank at any time t, $0 \leqslant t \leqslant 1.5$. [**6**]
Hint Use the call-put parity formula and the formula $\frac{\partial C}{\partial S} = \Phi(\omega)$, where C is the price of the call option.

(ii) Suppose that after 6 months the price of the share has grown to £35. Calculate the total value of the hedging portfolio in 6 months from now. How many shares should be in the portfolio and how much money should be deposited in the bank? [**8**]

Question 4 [19 marks].

(a) A random process Z_t is defined by

$$Z(t) = \int_0^t e^{-s} dW_s.$$

(i) Compute the variance of this process and state the distribution of $Z(t)$. **[4]**

(ii) Compute the expectation $\mathbb{E}\left(e^{Z(t)}\right)$. **[4]**

(b) A random process $V(t)$ is defined by

$$V(t) = \int_0^t \sqrt{s} e^{W_s} dW_s.$$

Compute the expectation and the variance of this process. **[5]**

(c) A random process S_t, $t \geqslant 0$, satisfies the following stochastic differential equation:

$$dS_t = aS_t dt + bS_t dW_t,$$

where a and b are real numbers. Define a new process Y_t by $Y_t = \frac{1}{S_t}$. Compute dY_t and prove that Y_t satisfies the following stochastic differential equation:

$$dY_t = AY_t dt + BY_t dW_t\,,$$

where A and B are functions of a and b. **[6]**

Question 5 [19 marks]. A model that is an extension of the Hall-White model for a variable interest rate r_t, $t \geqslant 0$, is described by the following stochastic differential equation:

$$dr_t = -a(r_t - \mu(t))dt + \sigma(t) dW_t,$$

where $a > 0$ is a constant, $\mu(t)$ and $\sigma(t)$ are strictly positive functions of t.

(a) Compute the differential of the function $U(t)$ defined by $U(t) = e^{at} r_t$. **[6]**

(b) Solve the equation for r_t with the initial value $r(0) = r_0$ **[7]**

(c) State the distribution of r_t and provide formulae for $\mathbb{E}(r_t)$ and $\mathrm{Var}(r_t)$. **[6]**

Question 6 [10 marks]. This question is about the **Merton model**.

Use the following notation.
$F(t)$, $0 \leqslant t \leqslant T$ for the total value of the corporate entity,
$E(0)$ for the equity of the corporate entity at time 0,
$B(0)$ for the capital the corporate entity raises by selling bonds at time 0,
L for the debt to the bondholders at time T.
The continuously compounded interest rate is r.

(a) Write down the equation which relates all the above quantities within the framework of the Merton model. **[3]**

(b) Prove that $B(0) \leqslant e^{-rT}L$. **[4]**

(c) A company's equity is £3 million. To develop a new product, the company needs another £2 million. To raise this money, the company issues zero coupon bonds with total value £2 million. It promises to repay the debt to bondholders in 2 years' time. The continuously compounded interest rate is $r = 0.05$, and the volatility $\sigma = 0,2$. Prove that L must be at least £2.2 million. **[3]**

End of Paper – An appendix of 1 page follows.

Table of the cumulative standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \quad \Phi(-x) = 1 - \Phi(x)$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

End of Appendix.