Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH6155: Financial Mathematics II

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

You have **24 hours** to complete and submit this assessment. When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about 2 hours to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: I. Goldsheid, P. Vergel

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1. The following convention is used in this paper. If $Y(t)$ is a random process then Y_t may be used to describe the same process; a similar convention applies to any other random process. In particular, both $W(t)$ and W_t denote the standard Wiener process.
2. $\tilde{\mathbb{E}}$ denotes the expectation over a risk-neutral probability.
3. Time involved in calculations should be expressed in years. E. g., 3 months should be converted into 0.25 years.
4. The precision of calculations should be to 3 decimal places.
5. You may use without proof the equality $\mathbb{E}(e^{bW_t}) = e^{\frac{b^2}{2}t}$, where b is any real number.

Question 1 [17 marks]. This question is about the Wiener process and the geometric Brownian motion.

(a) The random process $Y(t),\ t \geqslant 0$, is defined as follows:

$$Y(t) = t^{\frac{1}{3}} W(t^{\frac{1}{3}}),$$

where $W(t),\ t \geqslant 0$, is the standard Wiener process.

(i) Compute $\mathrm{Cov}(Y_t, Y_s)$ and derive from this result the expression for the variance of Y_t. **[7]**

(ii) Is $Y(t)$ a Wiener process? **[3]**

(b) Consider the geometric Brownian motion $S(t) = Se^{\mu t + \sigma W(t)}$. Compute the expectation of the product $S(t)S(s)$ for $s \geqslant t \geqslant 0$. **[7]**

Question 2 [9 marks]. This question is about the Arbitrage Theorem.

(a) State the Arbitrage Theorem. **[3]**

(b) A 6-sided die is rolled. The 6 possible outcomes are 1, 2, 3, 4, 5, 6.

You can bet on any outcome. If you bet £1 on i and the outcome is $j \neq i$, then you lose your pound. But if you bet £1 on i and the outcome is i, then you get back your pound and a reward of £u. For what value of u will this game be arbitrage-free?

Justify your answer. **[6]**

Question 3 [10 marks].

The price $S(t)$ of a share is driven by a geometric Brownian motion with parameters $S,\ \mu,\ \sigma$, that is $S(t) = Se^{\mu t+\sigma W(t)}$. Suppose that a proportional dividend on this share is paid continuously at rate $q > 0$ and is reinvested in the share. The continuously compounded interest rate is r. Compute the no-arbitrage price of a derivative with expiration time T and a payoff function

$$R(T) = \frac{2}{T}\int_{\frac{T}{2}}^{T} S\,(T/2)\,S(t)dt.$$

Question 4 [17 marks]. The price of a share follows the geometric Brownian motion with parameters $\mu = 0.2$ and $\sigma = 0.18$. Presently, the share's price is £38. Consider a call option having one year until its expiration time and having a strike price of £40. The continuously compounded interest rate is 5%.

(a) What is the risk-neutral price C of this call option? **[5]**

(b) Suppose now that you are the seller of this option. At time $t = 0$ you get £C from the buyer of the option, where C is the risk-neutral price of the option. You then have to design a hedging strategy which would allow you to meet your financial obligation in one year's time. Your portfolio should consist of two investments: you are allowed to buy the underlying shares and to deposit money in the bank.

(i) The price of the share evolves according to a geometric Brownian motion. State the formulae you will need to compute the number of shares in the portfolio and the capital deposited in the bank at any time $t,\ 0 \leqslant t \leqslant 1$. **[4]**

(ii) Suppose that after 9 months the price of the share has grown to £40. What should be the total value of your portfolio in 9 months from now? How many shares should be in the portfolio and how much money should be deposited in the bank? **[8]**

Question 5 [9 marks]. The price $S(t)$ of a share follows the geometric Brownian motion $S(t) = Se^{\mu t+\sigma W(t)}$. The price of the share at $t = 0$ is $S =$ £25 and $\mu = 0.1$. The continuously compounded interest rate is 8%. However, the volatility σ is not known.

(a) Explain the definition of implied volatility. **[3]**

(b) A European call option on the above share with the strike price $K =$ £23 and expiration time of 6 months has the market price £3.1.

Does the equation defining the implied volatility have a solution? **[6]**

Hint. You are not supposed to solve the equation defining the volatility to answer this question.

Question 6 [26 marks]. Consider the extension of the Vasicek model for a variable interest rate r_t, $t \geqslant 0$, which is described by the following stochastic differential equation:

$$dr_t = -a(r_t - \mu)dt + \sigma(t)dW_t,$$

where $a > 0$, $\mu > 0$ are constants and $\sigma(t)$, $t \geqslant 0$, is a strictly positive function of t.

(a) Compute the differential of the function $U(t)$ defined by $U(t) = e^{at}(r_t - \mu)$. **[6]**

(b) Solve the equation for r_t with the initial value $r(0) = r_0$ **[7]**

(c) State the distribution of r_t and provide formulae for $\mathbb{E}(r_t)$ and $\mathrm{Var}(r_t)$. **[6]**

(d) Suppose that $a = 2$, $\mu = 0.05$, and $\sigma(t) = 0.1e^{\frac{t}{4}}$. In other words, the interest rate is governed by the following stochastic differential equation:

$$dr_t = -2(r_t - 0.05)dt + 0.1e^{\frac{t}{4}}dW_t.$$

Given that $r_0 = 0.05$, what is the probability that after one year the interest rate will be less than 0.03? **[7]**

Question 7 [12 marks].
The following facts about a company are known:
1. At present, its total capital is £3 million.
2. It has just sold zero-coupon bonds with the total nominal value of £2 million which it promises to repay in 18 months from now.
3. The total capital $F(t)$ of the company follows the geometric Brownian motion with parameters $\mu = 0.15$ and $\sigma = 0.2$.
The continuously compounded annual interest rate $r = 6\%$.
Within the framework of the **Merton model**, establish the following.

(a) What is the total value of the shares of this company? **[5]**

(b) How much money has the company raised from the sale of the bonds? **[2]**

(c) What is the probability that the company would default on its promise to bond holders? **[5]**

End of Paper – An appendix of 1 page follows.

Table of the cumulative standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \quad \Phi(-x) = 1 - \Phi(x)$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

End of Appendix.