Queen Mary University of London

Main Examination period 2020 – May/June – Semester B
Online Alternative Assessments

MTH6155, MTH6155P: Financial Mathematics II

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment, you may use books, notes, and the internet. You may use calculators and computers, but you should show your working for any calculations you do. You must not seek or obtain help from anyone else.

At the start of your work, please **copy out and sign** the following declaration:

> I declare that my submission is entirely my own, and I have not sought or obtained help from anyone else.

All work should be **handwritten**, and should **include your student number**.

You have **24 hours** in which to complete and submit this assessment. When you have finished your work:

- scan your work, convert it to a **single PDF file** and upload this using the upload tool on the QMplus page for the module;
- e-mail a copy to maths@qmul.ac.uk with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are not expected to spend a long time working on this assessment. We expect you to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the assessment period, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: I. Goldsheid, S. Muirhead

1. The following convention is used in this paper. If $Y(t)$ is a random process then Y_t may be used to describe the same process; a similar convention applies to any other random process. In particular, both $W(t)$ and W_t denote the standard Wiener process.
2. $\tilde{\mathbb{E}}$ denotes the expectation over a risk-neutral probability.

Question 1 [28 marks]. This question is about the Wiener process, the Brownian motion, and the Geometric Brownian motion.

(a) Let W_t be the standard Wiener process. Prove that $\operatorname{Cov}(W_t, W_s) = \min(t, s)$. **[6]**

(b) Prove that if $W(t),\ t \geqslant 0$, is a Wiener process and $a > 0$ is a real number then also $Z(t) = \frac{1}{\sqrt{a}} W(at)$ is a Wiener process. **[10]**

(c) State the definition of a **Brownian motion** with the drift parameter μ and volatility parameter σ. **[3]**

(d) State the definition of a **geometric Brownian motion** $S(t)$ with initial value S, drift parameter μ, and volatility parameter σ. **[3]**

(e) Compute $\operatorname{Var}(S_t)$, where S_t is the geometric Brownian motion defined in (d). **[6]**

Hint. You may use without proof that $\mathbb{E}(\mathrm{e}^{bW_t}) = \mathrm{e}^{\frac{b^2}{2}t}$, where b is any real number.

Question 2 [22 marks]. Consider a share with price $S(t)$, $0 \leqslant t \leqslant T$. Suppose that a derivative on this share has a payoff function $R(T)$ (that is, the sum of £$R(T)$ is paid to the owner of the derivative at time T). Throughout the question, the continuously compounded interest rate is r.
Denote by C the no-arbitrage price of this derivative at time $t = 0$.

(a) Prove that
$$C = \mathrm{e}^{-rT} \tilde{\mathbb{E}}(R(T)),$$
where $\tilde{\mathbb{E}}$ is the expectation over the risk-neutral probability. **[7]**

(b) Suppose now that the price $S(t)$ of the share is driven by a geometric Brownian motion with parameters $S,\ \mu,\ \sigma$. Suppose also that a proportional dividend on this share is paid continuously at rate $q > 0$ and is reinvested in the share.

(i) State the parameters and the formula for the risk-neutral process $\tilde{S}(t)$ corresponding to $S(t)$. **[3]**

(ii) Compute the no-arbitrage price of a derivative with the payoff function $R(T) = \frac{1}{T} \int_0^T S_t^2 \mathrm{d}t$.
Hint. You may use, without proof, the following fact: if $Y(t)$ is a random process then
$$\mathbb{E}\left(\int_0^T Y(t) \mathrm{d}t \right) = \int_0^T \mathbb{E}\left(Y(t)\right) \mathrm{d}t.$$
[12]

Question 3 [13 marks]. The following notations are used in this question. If we deal with n shares then $S_j(t)$ is the price of the j^{th} share at time t (where $j = 1, 2, ..., n$). By $C_j(K_j, T)$ we denote the price of a European call option on the share with the price $S_j(t)$. (Here K_j is strike price and T is the expiration time of the j^{th} option.).

(a) State the definition of the index $I(t)$ of a portfolio consisting of n shares. **[4]**

(b) Let $C_I(K, T)$ be the price of a European call option on the index defined in (a). Let ω_j be the weights involved in the definition of the index. Prove that if $K = \sum_{j=1}^{n} \omega_j K_j$ then

$$C_I(K, T) \leqslant \sum_{j=1}^{n} \omega_j C_j(K_j, T).$$ **[9]**

Question 4 [18 marks].

(a) State the theorem which allows one to compute the distribution of the random variable $Z = \int_0^t f(s) dW_s$ and compute the parameters of this distribution when $f(s) = s^2$, $t = 2$. **[5]**

(b) Let $X = \int_0^1 W_s^2 \, dW_s$. Compute the variance of X. **[5]**

Hint. You may use without proof that $\mathbb{E}(W_t^{2k}) = \frac{(2k)!}{k! 2^k} t^k$ for any integer $k \geqslant 1$.

(c) State Ito's lemma which allows one to compute $dF(W_t)$, where $F : \mathbb{R} \mapsto \mathbb{R}$ is a function which has two continuous bounded derivatives. **[3]**

(d) Show that $\int_0^t W_s^3 \, dW_s = \frac{1}{4} W_t^4 - \frac{3}{2} \int_0^t W_s^2 \, ds$. **[5]**

Question 5 [10 marks].

The price of a share is described by the following stochastic differential equation:

$$dS_t = (a + c\sin t)S_t dt + \sigma S_t dW_t \quad \text{with} \quad S(0) = S_0, \tag{1}$$

where a, c, and σ are constants.

Solve this equation. **[10]**

Hint. To start, compute the stochastic differential $d\ln(S_t)$. Then integrate the equality you have obtained.

Question 6 [9 marks]. This question is concerned with the **Merton model**. Use the following notations:

$F(t)$, $0 \leqslant t \leqslant T$ for the total value of the corporate entity,

$E(0)$ for the equity of the corporate entity at time 0,

$B(0)$ for the capital the corporate entity raises by selling bonds at time 0,

L for the debt to the bondholders at time T.

The continuously compounded interest rate is r.

(a) State the definition of the Merton model. **[4]**

(b) Write down the payoff function for the shareholders and explain why, within the framework of the Merton model, the shareholders can be treated as having a European call option with the strike price L and the expiration time T. **[5]**

End of Paper.