Queen Mary
University of London

Main Examination period 2019

MTH6155, MTH6155P: Financial Mathematics II

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: I. Goldsheid, S. Muirhead

1. The following convention is used in this paper. If $Y(t)$ is a random process then Y_t may be used to describe the same process; a similar convention applies to any other random process. In particular, both $W(t)$ and W_t denote the standard Wiener process.
2. $\tilde{\mathbb{E}}$ denotes the expectation over a risk-neutral probability.

Question 1. [**20 marks**] This question is about the Wiener process.

(a) State the definition of a **Wiener process**. [**6**]

(b) Prove that if $W(t),\ t \geqslant 0$, is a Wiener process then also $Z(t) = W(t+s) - W(s)$ is a Wiener process. Here $s > 0$ is a fixed number. [**8**]

(c) Prove that for any real numbers $a,\ b$ and $s,\ t$ such that $0 \leqslant s < t$ the following equality holds:

$$\mathbb{E}\left(\mathrm{e}^{aW(s)}\mathrm{e}^{bW(t)}\right) = \mathrm{e}^{\frac{(a+b)^2}{2}s + \frac{b^2}{2}(t-s)}.$$ [**6**]

Hints. 1. Use the equality $\mathrm{e}^{aW(s)}\mathrm{e}^{bW(t)} = \mathrm{e}^{(a+b)W(s)}\mathrm{e}^{b(W(t)-W(s))}$ and the fact that the increments of the Wiener process are independent.

2. You may use, without proof, that $\mathbb{E}\left(\mathrm{e}^{\sigma W(t)}\right) = \mathrm{e}^{\frac{\sigma^2}{2}t}$ for any σ and any $t \geqslant 0$.

Question 2. [**24 marks**] Consider a share with price $S(t),\ 0 \leqslant t \leqslant T$. Suppose that a derivative on this share has a payoff function $R(T)$ (that is, the sum of £$R(T)$ is paid to the owner of the derivative at time T). The continuously compounded interest rate is r. Denote by C the no-arbitrage price of this derivative at time $t = 0$.

(a) Prove that

$$C = \mathrm{e}^{-rT}\tilde{\mathbb{E}}(R(T)),$$

where $\tilde{\mathbb{E}}$ is the expectation over the risk-neutral probability. [**7**]

(b) Suppose now that the price $S(t)$ of the share is driven by a geometric Brownian motion with parameters $S,\ \mu,\ \sigma$, that is $S(t) = S\mathrm{e}^{\mu t + \sigma W(t)}$.

(i) State the parameters and the formula for the risk-neutral process $\tilde{S}(t)$ corresponding to $S(t)$. [**3**]

(ii) Suppose that $R(T) = R(S(t_1), S(t_2), ..., S(t_n))$, where $0 \leqslant t_1 < t_2 < ... < t_n \leqslant T$. State the theorem which allows one to compute the no-arbitrage price of a derivative with this payoff function in terms of the expectation $\mathbb{E}$ and the risk-neutral process $\tilde{S}(t)$. [**4**]

(iii) Compute the no-arbitrage price of a derivative with the payoff function $R(T) = \sqrt{S(s)S(t)}$ where $0 < s < t \leqslant T$.
Hint Use the equality from Question 1(c). [**10**]

Question 3. **[17 marks]** Let $S(t)$, $0 \leqslant t \leqslant T$, be the price of a share. Suppose that a proportional dividend on this share is paid continuously at rate $q > 0$ and is reinvested in the share. The continuously compounded interest rate is r. Denote by C the price of a European call option, $\mathrm{Call}(K, T)$, and by P the price of a European put option $\mathrm{Put}(K, T)$ on this share.

(a) At time 0 a portfolio consists of one share.

(i) How many shares does this portfolio contain at time t? [**2**]

(ii) State the payoff function $R(T)$ of this portfolio and use it to prove that $\tilde{\mathbb{E}}(S(T)) = \mathrm{e}^{(r-q)T} S(0)$. [**7**]

(b) Prove that the following Call-Put parity relation holds:

$$C - P = \mathrm{e}^{-qT} S(0) - \mathrm{e}^{-rT} K.$$ [**8**]

Hint Make use of the result stated in Question 3(a)(ii) and the equality $x^+ - (-x)^+ = x$ (without proving it).

Question 4. **[12 marks]**

(a) State the distribution of the random variable $Z = \int_0^2 t^{\frac{3}{2}} dW_t$ and compute the parameters of this distribution. [**4**]

(b) Use Ito's lemma to show that $d(W_t^4) = 4W_t^3 dW_t + 6W_t^2 dt$. [**4**]

(c) Show that $\int_0^t W_s^3 \, dW_s = \frac{1}{4} W_t^4 - \frac{3}{2} \int_0^t W_s^2 ds$. [**4**]

Hint Use the relation stated in Question 4(b) or any other method you may know.

Question 5. [14 marks] The price $S(t)$ of a share satisfies the following stochastic differential equation:

$$dS_t = aS_t dt + \sigma S_t dW_t \quad \text{with} \quad S(0) = S_0,$$

where a and σ are constants.

(a) Use the chain rule to compute $d \ln S(t)$. **[7]**

(b) Use the result obtained in (a) to compute $\ln S(t)$. Thus find $S(t)$. **[7]**

Question 6. [13 marks] This question is concerned with the **Merton model**. Use the following notations:
$F(t)$, $0 \leqslant t \leqslant T$ for the total value of the corporate entity,
$E(0)$ for the equity of the corporate entity at time 0,
$B(0)$ for the capital the corporate entity raises by selling bonds at time 0,
L for the debt to the bondholders at time T.
The continuously compounded interest rate is r.

(a) State the definition of the Merton model. **[4]**

(b) Write down the payoff function for the shareholders and explain why, within the framework of the Merton model, the shareholders can be treated as having a European call option Call(L, T). **[5]**

(c) Recall that the payoff the bondholders receive at time T is given by

$$R_{bh}(T) = \min[F(T),\, L].$$

Use this formula to prove that $B(0) \leqslant e^{-rT} L$. **[4]**

End of Paper.