Queen Mary University of London

Main Examination period 2018

MTH6151: Partial Differential Equations

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: Dr. Juan A. Valiente Kroon

Throughout we only consider partial differential equations in two independent variables (x, y) or (x, t).

Question 1. [20 marks]

(a) Explain, in a few words, what is a **characteristic curve** of a first order linear partial differential equation. **[4]**

(b) Determine whether the following partial differential equations are linear or non-linear. Also, say whether they are homogeneous or inhomogeneous:

(i) $U_x + UU_{yy} - U = 0,$ **[2]**

(ii) $5U_{yy} - yU_x = 0.$ **[2]**

(c) Using the method of characteristics, or otherwise, solve the equation

$$U_t - 3U_x = 0$$

subject to the condition

$$U(x, 0) = \cos x.$$ **[5]**

(d) Solve the equation

$$yU_x + xU_y = 0$$

under the condition

$$U(0, y) = e^{-y^2}.$$ **[7]**

Question 2. [12 marks]

(a) Classify, according to type (hyperbolic, elliptic, parabolic), the equations:

(i) $U_{xx} + 2U_{xy} + U_{yy} - U_x + U = 0.$ **[2]**

(ii) $2U_{xy} + U_y + U_x = 0.$ **[2]**

In each case provide a brief (one line) justification.

(b) Use the coordinate transformation

$$x' = x,$$
$$y' = -\frac{3}{2}x + y$$

to find the general solution to the second order partial differential equation

$$4U_{xx} + 12U_{xy} + 9U_{yy} = 0.$$ **[8]**

Question 3. [28 marks]
Throughout this question, consider the following problem for the wave equation:

$$\begin{aligned} &U_{tt} - c^2 U_{xx} = 0, \qquad x \in [0, L], \quad t \geq 0, \\ &U(x, 0) = f(x), \qquad U_t(x, 0) = g(x), \\ &U(0, t) = 0, \qquad U(L, t) = 0. \end{aligned}$$

(a) Following the method of separation of variables consider solutions of the form

$$U(x, t) = X(x)T(t)$$

where X and T are functions of a single argument. Show that X and T satisfy the ordinary differential equations

$$\begin{aligned} X'' &= -kX, \\ T'' &= -c^2 kT, \end{aligned}$$

for some constant k. Moreover, show that

$$X(0) = X(L) = 0.$$ **[6]**

(b) Show that the constant k obtained in (a) must be positive if $X(x)$ is not identically 0 for $x \in [0, L]$. **[6]**

(c) Find the general solution to the ordinary differential equations in (a). **[4]**

(d) Use the conditions $X(0) = X(L) = 0$ to determine the value of k and show that the solutions X obtained in (c) must be of the form

$$X(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots$$ **[4]**

(e) Use the **Principle of Superposition** to find the general solution to the wave equation on the interval $[0, L]$ with the prescribed boundary conditions. **[4]**

(f) Assuming that the general solution to the problem can be written as

$$U(x, t) = \sum_{n=1}^{\infty} \left(a_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right) \right)$$

where a_n, b_n are constants, find the particular solution corresponding to the initial data

$$\begin{aligned} &U(x, 0) = 0, \\ &U_t(x, 0) = \sin\left(\frac{2\pi x}{L}\right) + 3 \sin\left(\frac{3\pi x}{L}\right). \end{aligned}$$

[4]

Question 4. [20 marks]

(a) Show that the series

$$U(r,\theta) = \sum_{m=1}^{\infty} \left(A_m r^m + \frac{B_m}{r^m} \right) (\cos m\theta + \sin m\theta)$$

where A_m and B_m are constants is a solution to the Laplace equation

$$\Delta U = 0$$

in polar coordinates. Ignore questions of convergence of the series. **[6]**

(b) Show that if $U(x,y)$ is a solution to the Laplace equation

$$\Delta U = U_{xx} + U_{yy} = 0,$$

then

$$V(x,y) \equiv \frac{\partial U}{\partial x}$$

is also a solution to the Laplace equation. **[5]**

(c) Explain what is the **Principle of the Maximum** for the Laplace equation. **[4]**

(d) Suppose that U is a harmonic function on the disk $\Omega = \{(r,\theta) \,:\, r \leq 2\}$ and that

$$U(2,\theta) = 3\sin\theta + 1.$$

Find the maximum and minimum values of U on the disk Ω. **[5]**

Question 5. [20 marks]

(a) Explain in a few words what is the **heat kernel** and what is its relevance in the study of the heat equation. **[4]**

(b) Show that if $U(x,t)$ is a solution to the heat equation

$$U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}$$

then

$$V(x,t) = \int_{-\infty}^{\infty} U(x-y,t)g(y)dy$$

is also a solution to the heat equation for any function g. Ignore any issues about the convergence of the integral. **[6]**

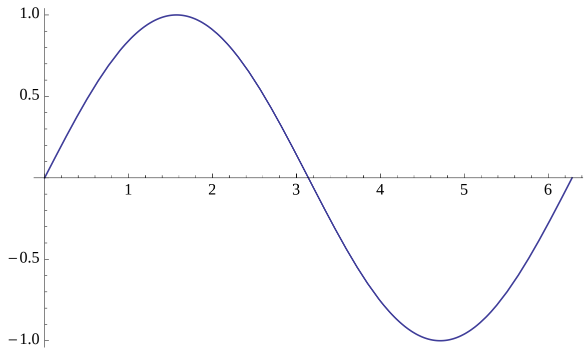
(c) Describe in qualitative terms the behaviour of the solution to the heat equation on an interval

$$U_t = \varkappa U_{xx}, \qquad x \in [0, 2\pi],$$

with initial data

$$U(x,0) = f(x)$$

where $f(x)$ has the form



and

$$U(0,t) = U(2\pi,t) = 0.$$

What do you expect to be the limit of $U(x,t)$ as $t \to \infty$? No proof or calculations are required. You may draw a plot of the solution at various instants of time to explain your answer. **[6]**

(d) What is the main difference one encounters when solving the heat equation on an interval by means of the method of separation of variables compared to the same procedure for the wave equation? What is the consequence of this difference in the behaviour of the solutions to the two equations? **[4]**

End of Paper – An appendix of 1 page follows.

The Laplacian in polar coordinates

The expression for the Laplacian for a function U on $\mathbb{R}^2$ in standard spherical coordinates (r, θ) is given by

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

Orthogonality properties of the sine function

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L/2 & \text{for} \quad n = m \\ 0 & \text{for} \quad n \neq m \end{cases} .$$

Gaussian integral

$$\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}.$$

End of Appendix.