Queen Mary University of London

M. Sci. Examination by course unit 2014

MTH744U Dynamical Systems

Duration: 3 hours

Date and time: 19 May 2014, 14.30–17.30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

This is an OPEN BOOK exam. Permitted are printed material, handwritten notes, and photocopies of any kind. Sharing material with other students is not permitted.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important note: the Academic Regulations state that possession of unauthorized material at any time by a student who is under examination conditions is an assessment offence and can lead to expulsion from QMUL.

Please check now to ensure you do not have any mobile phones or unauthorised electronic devices on your person. If you have any, then raise your hand and give them to an invigilator immediately. Be aware that if you are found to have hidden unauthorised devices elsewhere, including toilets and cloakrooms, they will be treated as being found in your possession. Disruption caused by mobile phones is also an examination offence.

Exam papers must not be removed from the examination room.

Examiner(s): F. Vivaldi

Question 1 [32 marks]

(Recall that $\dfrac{d\arctan(x)}{dx} = \dfrac{1}{1+x^2}$.)

Consider the differential equation on the line

$$\dot{x} = v(x) = \lambda x + \mu \arctan(x) \tag{1}$$

where λ and μ are real parameters, with $\mu > 0$.

(a) Show that the point 0 is a fixed point. Determine the parameter values for which 0 is a stable fixed point. Determine the bifurcation curve in the (λ, μ)-plane which marks the boundary of stability. [6 marks]

(b) Determine the region of parameters where there are two additional fixed points $\pm x^*$, where $x^* > 0$. Justify your answer. [6 marks]

(c) Use a Taylor approximation to compute an approximate vector field $\tilde{v}$ such that

$$v(x) = \tilde{v}(x) + O(x^4) \qquad \text{as} \quad x \to 0.$$

Hence determine an approximate expression for the fixed point x^* defined in part (b). [8 marks]

(d) Now fix μ and vary λ. What type of bifurcation takes place at the critical parameter? Identify its normal form by verifying the appropriate transversality conditions. [6 marks]

(e) For fixed μ, determine the linear change of co-ordinates that brings the approximate system $\dot{x} = \tilde{v}(x)$ of part (c) to its normal form at bifurcation; hence sketch the bifurcation diagram in the (λ, x)-plane. [6 marks]

Question 2 [26 marks]

Consider the dynamical system in the plane:

$$\begin{aligned} \dot{x} &= ay \\ \dot{y} &= y + x(1-x) \end{aligned} \tag{2}$$

where a is a real parameter, with $a \geqslant 0$.

(a) Determine all fixed points and their Jacobian matrices, as functions of the parameter a. [6 marks]

(b) For each fixed point, determine the linear stability, the eigenvectors and the Jordan canonical form of the Jacobian, as a function of a. [12 marks]

(c) Let $a = 2$ in equation (2). Draw the nullclines and hence sketch the phase portrait. Indicate the stable and unstable manifold for any saddles. [8 marks]

Question 3 [20 marks]

Consider the second-order differential equation on the line

$$\ddot{x} + 3x^2 = 1. \tag{3}$$

(a) Transform equation (3) into a system of two first-order differential equations in the plane. [2 marks]

(b) Show that this system is reversible. Write the equation of the energy, and verify explicitly that it is a constant of the motion. [8 marks]

(c) Sketch the phase portrait, as level sets of the energy; show that this system has a homoclinic orbit and write an equation for this orbit. [10 marks]

Question 4 [22 marks]

(a) Consider the planar dynamical system, in polar coordinates

$$\dot{r} = f(r), \qquad \dot{\theta} = 1 \tag{4}$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is a differentiable function.

i) Give an example of a function f for which the system (4) has:

1) a single limit cycle, which is neither stable nor unstable;
2) infinitely many limit cycles.

Justify your answers; in 2) comment on the stability of the cycles. [8 marks]

ii) Prove (from the definition) that for all initial conditions $z_0 = (r_0, \theta_0)$ with $r_0 \neq 0$, the ω-limit set $\omega(z_0)$ of the system (4) with $f(r) = r(1-r)$ is the unit circle $r = 1$. [8 marks]

(b) For all initial conditions $z_0 = (r_0, \theta_0)$ list explicitly the ω-limit sets $\omega(z_0)$ of the system

$$\dot{r} = r(1-r); \qquad \dot{\theta} = 1 + \cos(2\theta) + (r-1)^2.$$

Justify your answers. Does this system have homoclinic/heteroclinic orbits? (No formal proof is required.) [6 marks]

End of Paper