Main Examination period 2023 – May/June – Semester B

MTH743U / MTH743P: Complex Systems

Duration: 3 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **3 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

You are allowed to bring **three A4 sheets of paper** as notes for the exam.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: O. Jenkinson, V. Latora

Throughout this exam, I denotes the closed unit interval $[0, 1]$.

Question 1 [40 marks].

(a) What does it mean to say that a map $f : I \to I$ is a **Markov map**? **[3]**

(b) How is the **transition matrix** for a Markov map defined? **[3]**

(c) What does it mean to say that a Markov map is **expanding**? **[3]**

(d) Give an example of a Markov map that is not expanding. **[3]**

(e) Give an example of any expanding Markov map f whose transition matrix is equal to

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

being careful to specify the associated partition. Sketch the graph of f, being careful to mark any fixed points of f in your sketch, together with the numerical values of these fixed points. **[5]**

(f) Compute the topological entropy of your map f from (e) above. **[5]**

(g) Suppose that $f : I \to I$ is continuous, and $a \in (0, 1)$. Suppose that f is a piecewise-linear Markov map with Markov partition $\{I_0, I_1\}$, where $I_0 = [0, a]$, $I_1 = [a, 1]$. Show that if the corresponding transition matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

then f is expanding. **[5]**

(h) For a map f as in (g) above, how many period-3 points are there? Determine all of these period-3 points. **[5]**

(i) For a map f as in (g) above, how many period-10 points are there? **[5]**

(j) Let N_{30} denote the number of period-30 points for a map f as in (g) above. Determine the integer k such that $10^k < N_{30} < 10^{k+1}$. **[3]**

Question 2 [30 marks]. Define $f : I \to I$ by

$$f(x) = \begin{cases} \sqrt{2x^2 + x} & \text{if } x \in [0, 1/2) \\ \sqrt{2x^2 - x} & \text{if } x \in [1/2, 1]\,. \end{cases}$$

(a) Sketch the graph of f, show that f is a Markov map with respect to the partition $\{[0, 1/2], [1/2, 1]\}$, and show that f is expanding. **[10]**

(b) Determine all the period-2 points of f. **[5]**

(c) Determine formulae for the inverse branches $\varphi_0 : I \to [0, 1/2]$ and $\varphi_1 : I \to [1/2, 1]$ of f, and compute formulae for their derivatives φ_0' and φ_1'. **[5]**

(d) Write down an explicit form of the Frobenius-Perron equation for f, and use this to determine the invariant density ϱ for f, assuming that ϱ is of the form $\varrho(x) = ax + b$ for some $a, b \in \mathbb{R}$. **[5]**

(e) Assuming ergodicity, compute the value $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f^n(x)$ for Lebesgue-typical points $x \in I$. **[5]**

Question 3 [30 marks]. Suppose $f : I \to I$ is defined by

$$f(x) = \begin{cases} 4x & \text{if } 0 \le x \le 1/4, \\ 2x - 1/4 & \text{if } 1/4 < x \le 1/2, \\ 2x - 1 & \text{if } 1/2 < x \le 3/4, \\ 3x - 2 & \text{if } 3/4 < x \le 1. \end{cases}$$

(a) Sketch the graph of f, show that f is a Markov map with respect to the partition $\{[0, 1/4], [1/4, 1/2], [1/2, 3/4], [3/4, 1]\}$, and write down the corresponding transition matrix. **[10]**

(b) Determine the transfer matrix for f. Use the transfer matrix to determine the invariant density for f, and sketch the graph of this invariant density. **[10]**

(c) Assuming ergodicity, determine whether or not there exists a sub-interval of length 1/2 that is entered with frequency $> 3/4$ by Lebesgue-typical orbits (i.e. determine whether or not there exists c such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_{[c,c+1/2]}(f^n(x)) > 3/4$$

for Lebesgue-typical points $x \in I$). Justify your answer. **[5]**

(d) Assuming ergodicity, if $c, l \ge 0$ are such that $[c, c + l]$ is a sub-interval of I then let $\beta(c, l) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_{[c,c+l]}(f^n(x))$ for Lebesgue-typical points $x \in I$, and let $\alpha(l) = \sup\{\beta(c, l) : [c, c + l] \subseteq I\}$ for all $l \in [0, 1]$.

Find a formula for the function $\alpha : [0, 1] \to \mathbb{R}$, and sketch the graph of α. **[5]**

End of Paper.