Queen Mary
University of London

# MTH743U: Complex Systems

## Duration: 3 hours

## Date and time: 9 June 2016, 10:00–13:00

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**Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.**

**You should attempt ALL questions. Marks awarded are shown next to the questions.**

**Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.**

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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**Exam papers must not be removed from the examination room.**

**Examiner(s): V. Latora**

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**Question 1 (35 marks).**
Consider the map $f : [0,1] \to [0,1]$ defined as:

$$f(x) = \begin{cases} (6+3x)/8 & \text{if} \quad 0 \le x \le 2/3 \\ 3(1-x) & \text{if} \quad 2/3 < x \le 1 \end{cases}$$

(a) Sketch the graph of the map $f$ and find its fixed points. **[8]**

(b) Show that the partition $\{I_0, I_1, I_2\}$ with $I_0 = [0, 2/3]$, $I_1 = [2/3, 3/4]$ and $I_2 = [3/4, 1]$ is a Markov partition, and that $f$ is a piecewise linear expanding Markov map. **[9]**

(c) Find the corresponding topological transition matrix and determine all admissible periodic symbol sequences of period $p = 2$. Find the number of periodic points of period $p = 2$ of the map. **[9]**

(d) Determine the transfer matrix of the map $f$ and write down its invariant density. **[5]**

(e) Assuming ergodicity, evaluate the Lyapunov exponent of the map $f$. **[4]**

**Question 2 (30 marks).**
Consider the map $f:[0,1]\to[0,1]$ defined by:

$$f(x) = \begin{cases} \frac{1}{2}\left[\sqrt{16x+1}-1\right] & \text{if} \quad 0 \le x \le 1/2 \\ \\ \frac{1}{2}\left[\sqrt{17-16x}-1\right] & \text{if} \quad 1/2 < x \le 1 \end{cases} .$$

**a)** Find the two invertible branches of the map and write down the two inverse functions. [**10 marks**]

**b)** Write down the Frobenius-Perron equation of the map. [**6 marks**]

**c)** Consider the function

$$\rho(x) = c(x+1/2)$$

and determine the value of the normalisation constant $c$ such that $\rho(x)$ is the density of a probability measure on $[0,1]$. [**4 marks**]

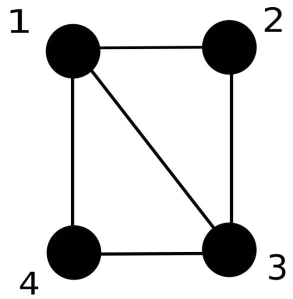
**d)** Show that $\rho(x)$ is a solution of the Frobenius-Perron equation of the map $f$. [**6 marks**]

**e)** Assuming that the density $\rho(x)$ gives rise to an ergodic invariant measure, compute the Lyapunov exponent of the map $f$. [**4 marks**]

**Question 3 (35 marks).**
Consider the following equations of motion:

$$\dot{x}_i(t) = f(x_i(t)) + \sigma \sum_j G_{ij} h(x_j(t))$$

describing the dynamics of a coupled network. Here, $x_i(t)$ denotes the state at node $i$, $f(x) = x - x^3$ governs the local dynamics at a node, $h(x) = x$ determines the coupling, $\sigma \in \mathbb{R}$ denotes the coupling strength, and $G_{ij}$ is the Laplacian of the underlying graph. The network consists of $N = 4$ nodes connected as in the graph below:




**a)** Compute the average node degree, the degree distribution, the average distance between nodes, and the diameter of the graph. **[9 marks]**

**b)** Determine the time-independent synchronised states. **[5 marks]**

**c)** For each of the time-independent synchronised states, compute the master stability function. **[9 marks]**

**d)** Write down the Laplacian of the network, and show that it has eigenvalues $\lambda = 0, 2, 4, 4$. **[7 marks]**

**e)** For each synchronised state find the values of the coupling strength $\sigma$ such that the state is transversely stable. **[5 marks]**

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**End of Paper.**