Queen Mary University of London

Main Examination period 2023 – January – Semester A

MTH742P: Advanced Combinatorics (resit paper)

Duration: 3 hours

The exam is intended to be completed within **3 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: Robert Johnson and John Bray

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In this paper graphs do not have loops or multiple edges.

Question 1 [25 marks].

(a) Define the **Turán density** $\pi(F)$ of a finite graph F. **[3]**

(b) State (without proof) a useful formula relating $\pi(F)$ to another graph parameter of F, defining any terms you use. **[3]**

(c) For $t \geqslant 2$, let A_t be the graph on t^2 vertices defined by

- $V(A_t) = \{(x, y) : x, y \in \{1, 2, \ldots, t\}\}$
- Distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $x_1 = x_2$ or $y_1 = y_2$.

Use part (b) to determine $\pi(A_t)$ for all $t \geqslant 2$. **[5]**

(d) For $t \geqslant 2$, let B_t be the graph on t^2 vertices defined by

- $V(B_t) = \{(x, y) : x, y \in \{1, 2, \ldots, t\}\}$
- (x_1, y_1) and (x_2, y_2) adjacent if and only if either (i) $x_1 = x_2$ and $|y_1 - y_2| = 1$ or (ii) $y_1 = y_2$ and $|x_1 - x_2| = 1$.

Use part (b) to determine $\pi(B_t)$ for all $t \geqslant 2$. **[5]**

(e) For each of the following conditions either give an example of a finite graph F satisying it or explain why no such graph exists: **[9]**

(i) $\pi(F) = \frac{\sqrt{2}}{2}$

(ii) F has 7 vertices and $\pi(F) = 0.9$

(iii) F is K_4-free and $\pi(F) = \pi(K_4)$

Question 2 [25 marks].

(a) State the maximum number of edges in a triangle-free graph on n vertices, and describe the family of graphs attaining it. **[5]**

(b) Suppose that n is even and $0 \leqslant r \leqslant n - 1$. For which values of r does there exist an r-regular triangle-free graph on n vertices. Justify your answer. **[7]**

(c) Decide whether the following statement is true or false, giving a proof or counterexample as apropriate: For any triangle-free graph G with fewer edges than the maximum you gave in part (a), it is possible to add an edge to G without creating a triangle. **[6]**

(d) Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$ satisfy that there are no distinct i, j, k with $x_i + x_j + x_k = 3$. What is the maximum number of pairs (i, j) with $1 \leqslant i < j \leqslant n$ satisfying $x_i + x_j = 2$. Prove your answer. **[7]**

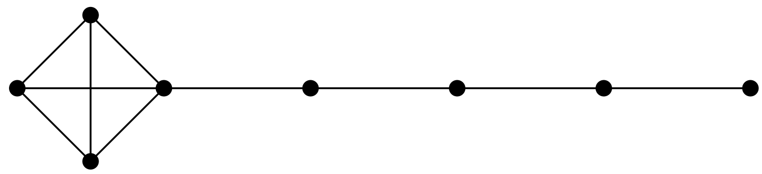
Question 3 [25 marks]. This question is about subgraphs of the random graph $G_{n,p}$.

(a) Let X be the random variable which counts the number of copies of K_4 in $G_{n,p}$. Determine a formula for $\mathbb{E}(X)$ and show that if $p = n^{-c}$ for some $c > 2/3$ then $\lim_{n\to\infty} \mathbb{E}(X) = 0$. **[5]**

(b) Deduce that when $p = n^{-c}$ for some $c > 2/3$ we have

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \text{ is } K_4\text{-free}) = 1.$$ **[5]**

(c) Let H be the graph below. Let Y be the random variable which counts the number of copies of H in $G_{n,p}$.



Show that if $p = n^{-c}$ for some $c > 4/5$ then $\lim_{n\to\infty} \mathbb{E}(Y) = 0$. **[5]**

(d) Find a p significantly larger than $n^{-4/5}$ for which $\lim_{n\to\infty} \mathbb{P}(G_{n,p} \text{ is } H\text{-free}) = 1$. **[5]**

(e) Construct a graph F satisfying both of the following:

- When $p = n^{-1/10}$ we have $\lim_{n\to\infty} \mathbb{P}(G_{n,p} \text{ is } F\text{-free}) = 1$
- When $p = n^{-9/10}$ we have $\lim_{n\to\infty} \mathbb{E}(\text{the number of copies of } F \text{ in } G_{n,p}) \not\to 0$.

Justify why your example works. **[5]**

Question 4 [25 marks].

(a) Show that the number of copies of $K_{1,t}$ (the star with t leaves) in an n-vertex graph with $e \geqslant \frac{tn}{2}$ edges is at least

$$\frac{n}{t!}\left(\frac{2e}{n} - t\right)^t$$

[7]

(b) Suppose that $\frac{2e}{n}$ is an integer. Which graphs on n vertices with e edges contain the fewest copies of $K_{1,t}$? **[5]**

(c) Use the $t = 2$ case of part (a) to give an upper bound on $\mathrm{ex}(n, K_{2,4})$. **[5]**

(d) Use the $t = 4$ case of part (a) to give an upper bound on $\mathrm{ex}(n, K_{2,4})$. **[5]**

(e) Which of parts (c) and (d) is a stronger bound for large n? **[3]**

Question 5 [25 marks].

(a) State and prove Dirac's Theorem on Hamilton cycles in graphs. [**5**]

(b) Let $X = \{1, 2, \ldots, n\}$. The **Kneser graph** $K(n, r)$ is defined by

$$V(K(n,r)) = \{A \subseteq X : |A| = r\}$$

with A, B adjacent if $A \cap B = \emptyset$.

(i) Show that $K(n, 2)$ is Hamiltonian for all $n \geqslant 8$. [**6**]

(ii) Find a necessary and sufficient condition on n and r for $K(n, r)$ to be triangle-free. [**5**]

(c) For each of the following conditions either give an example of a graph satisfying it or prove that no such graph can exist: [**9**]

(i) G is a bipartite graph on an odd number of vertices with a Hamilton cycle.

(ii) G is a regular graph on an even number $n \geqslant 4$ of vertices and both G and its complement $\overline{G}$ are non-Hamiltonian.

(iii) G is a 4-regular Hamiltonian graph on 100 vertices with the property that deleting any vertex (and its incident edges) from it gives a Hamiltonian graph.

End of Paper.